# Probabilistic Counting

# Counting with a coin with head probability p

- Let n = 3. Tabular all possible counter values (c) in the table.

| n=0 | n=1 | n=2 | n=3 | probability |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $(1-p)^3 * p^0$ |
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$$E[c] = ?$$

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$$E[c] = \sum_{c=0}^{3} c \cdot \left( \binom{3}{c} (1-p)^{3-c} \cdot p^c \right)$$

- The formula is based on the observation that "probability" values follows the Binom(*n*=3, *p*)
  - c can be deemed as # of heads!
- Hence E[c] is the expected value of this Binom distribution, which is *np = 3p.*
- Hence, we verified: E[c]/p = n

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$$P(n \mid c) = ?$$

- What does this probability mean?
  - Distinguish it with $P(c \mid n)$

$$E[c] = \sum_{c=0}^{3} c \cdot \left(\binom{3}{c}(1-p)^{3-c} \cdot p^c\right)$$

$$E[c \mid n] = p \cdot n$$

- P(c | n=3): the probability of seeing a counter value of c after processing n=3 items.
- P(n | c=2): the probability of n when I “peeked” at the counter and see a value of c=2, and I want to know the (distribution of) n at that time.

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$$P(n \mid c) = ?$$

Hard to be computed "objectively" as we need P(n) !
➔ Assume P(n) = 1/n ← also known as "uninformative prior"

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$$P(n \mid c) = ?$$

$$P(c \mid n) = \binom{n}{c} p^c (1-p)^{n-c}$$

$$P(n) = \frac{1}{n}$$

$$P(n \mid c) = \frac{\binom{n}{c} p^c (1-p)^{n-c} \frac{1}{n}}{\sum_{m=c}^{\infty} \binom{m}{c} p^c (1-p)^{m-c} \frac{1}{m}}$$

n ≥ c

- Note that the formula and computation for the full-fledged Morris' counting can be derived in the same way, but without a simplistic formula.

$$\text{for } p = \frac{1}{2}, P(n \mid 2) = (n-1)\left(\frac{1}{2}\right)^n, \quad n \geq 2.$$

# Final Note

- Let n = 3. Tabular all possible counter values (c) in the table.

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$$P(n \mid c) = ?$$

- Note that the formula and computation for the full-fledged Morris' counting can be derived in the same way, but without a simple formula.
  - p is changing over n and over each path
- If we only consider the final probability P(c|n) in Morri's counting, it will be a "fixed" valid and sophisticated distribution
  - We did not explicitly specify (aka. parameterized) the distribution, but instead, we specify the ***stochastic process*** that results in this distribution.