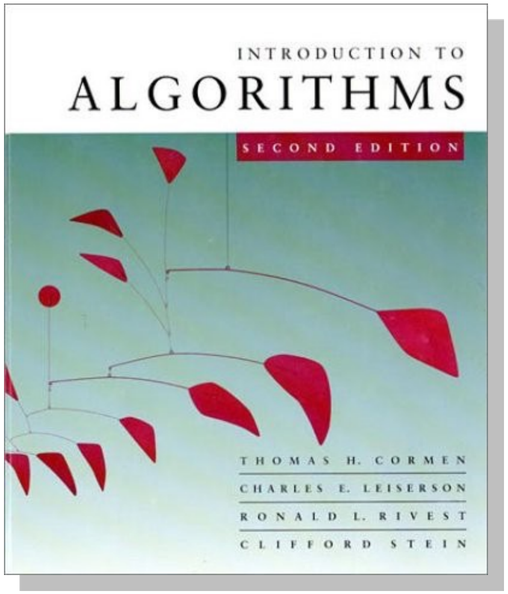
# *Introduction to Algorithms*

## 6.046J/18.401J




## LECTURE 4

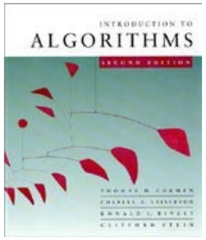
**Quicksort**

- Divide and conquer
- Partitioning
- Worst-case analysis
- Intuition
- Randomized quicksort
- Analysis

**Prof. Charles E. Leiserson**

# Quicksort

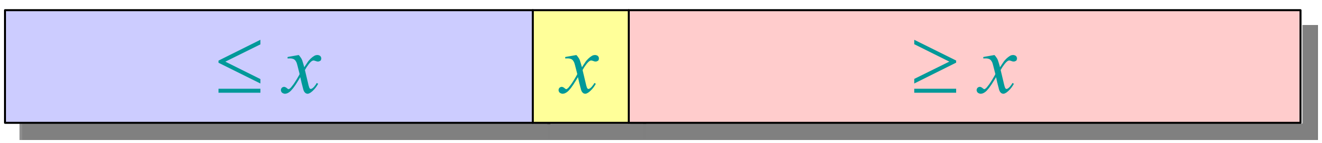
- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts “in place” (like insertion sort, but not like merge sort).
- Very practical (with tuning).




# Divide and conquer

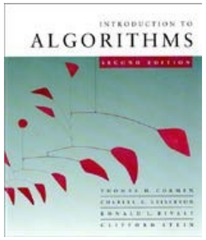
Quicksort an $n$-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** $x$ such that elements in lower subarray $\leq x \leq$ elements in upper subarray.




2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

**Key:** *Linear-time partitioning subroutine.*




# Qsort + Naïve Partitioning

QUICKSORT($A$)

  $A_{left}$, $A[q]$, $A_{right}$ $\leftarrow$ PARTITION($A$, $p$, $r$)

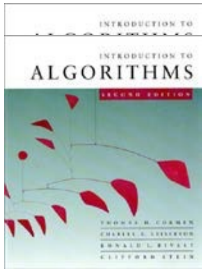
  QUICKSORT($A_{left}$)

  Output($A[q]$)

  QUICKSORT($A_{right}$)

**Initial call:** QUICKSORT($A$)

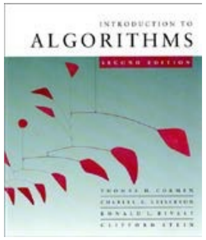
Space overhead:




# Quicksort (in-place)

QUICKSORT$(A, p, r)$
    **if** $p < r$
        **then** $q \leftarrow$ PARTITION$(A, p, r)$
                QUICKSORT$(A, p, q-1)$
                QUICKSORT$(A, q+1, r)$

**Initial call:** QUICKSORT$(A, 1, n)$

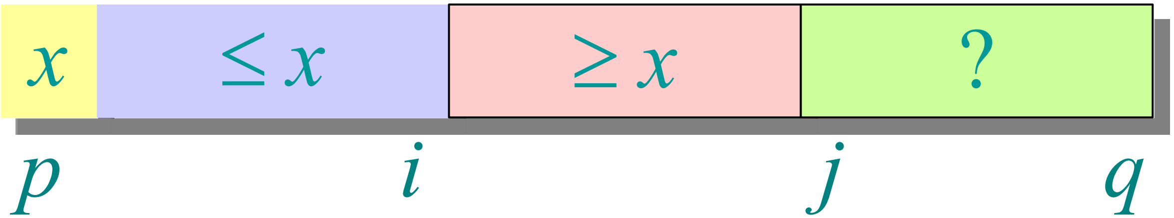


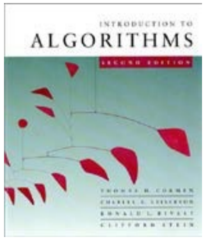

# Partitioning subroutine

PARTITION$(A, p, q)$ ⊳ $A[p \, . \, . \, q]$
    $x \leftarrow A[p]$ ⊳ pivot $= A[p]$
    $i \leftarrow p$
    **for** $j \leftarrow p + 1$ **to** $q$
        **do if** $A[j] \leq x$
            **then** $i \leftarrow i + 1$
                exchange $A[i] \leftrightarrow A[j]$
    exchange $A[p] \leftrightarrow A[i]$
    **return** $i$

Running time $= O(n)$ for $n$ elements.

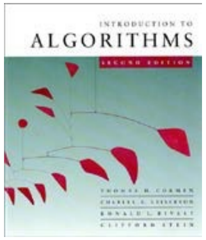
***Invariant:***






# Example of partitioning

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |
| $i$ | $j$ | | | | | | |




# Example of partitioning

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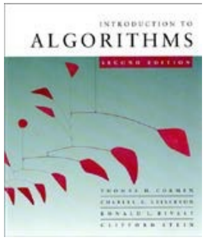
$i$ $\longrightarrow j$

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$i$ $j$

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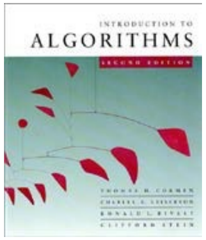
$i$ $\longrightarrow j$

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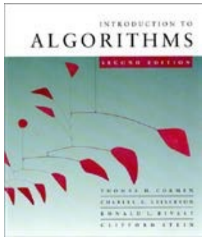
$\longrightarrow i$ $\qquad\qquad j$

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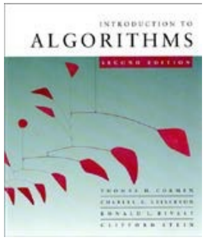


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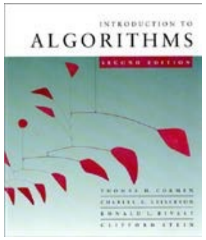
$i$ $j$

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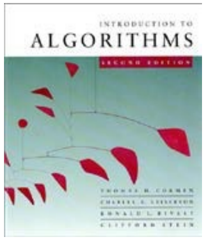
| | | | $i$ | | | | $j$ |

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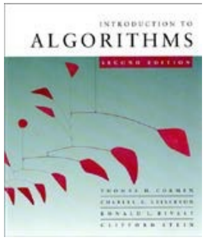
$i$ $\qquad\qquad \bullet\!\longrightarrow j$

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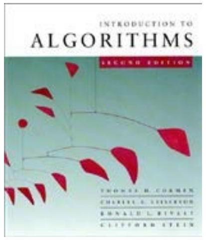
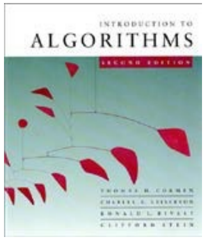
| 6 | 5 | 3 | 10 | 8 | 13 | 2 | 11 |
| 6 | 5 | 3 | 2 | 8 | 13 | 10 | 11 |
| 2 | 5 | 3 | 6 | 8 | 13 | 10 | 11 |
| | | | $i$ | | | | |




# Pseudocode for quicksort

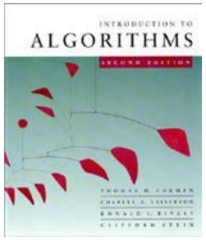
QUICKSORT$(A, p, r)$
    **if** $p < r$
        **then** $q \leftarrow$ PARTITION$(A, p, r)$
            QUICKSORT$(A, p, q{-}1)$
            QUICKSORT$(A, q{+}1, r)$

**Initial call:** QUICKSORT$(A, 1, n)$




# Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of $n$ elements.




# Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textit{(arithmetic series)}
\end{aligned}
$$

# Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

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$$T(n) = T(0) + T(n\text{–}1) + cn$$

$T(n)$

# Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

$$\begin{array}{ccc} & cn & \\ \swarrow & & \searrow \\ T(0) & & T(n-1) \end{array}$$

# Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

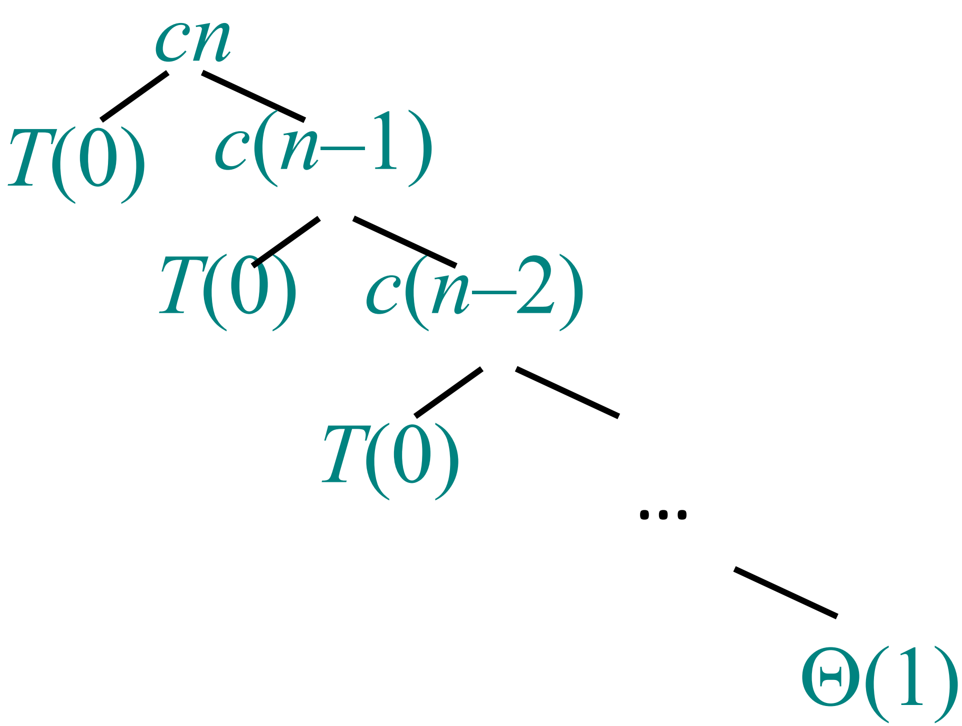
$$cn$$

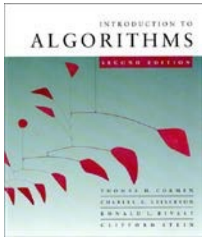
$$T(0) \qquad c(n-1)$$

$$T(0) \qquad T(n-2)$$

# Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$







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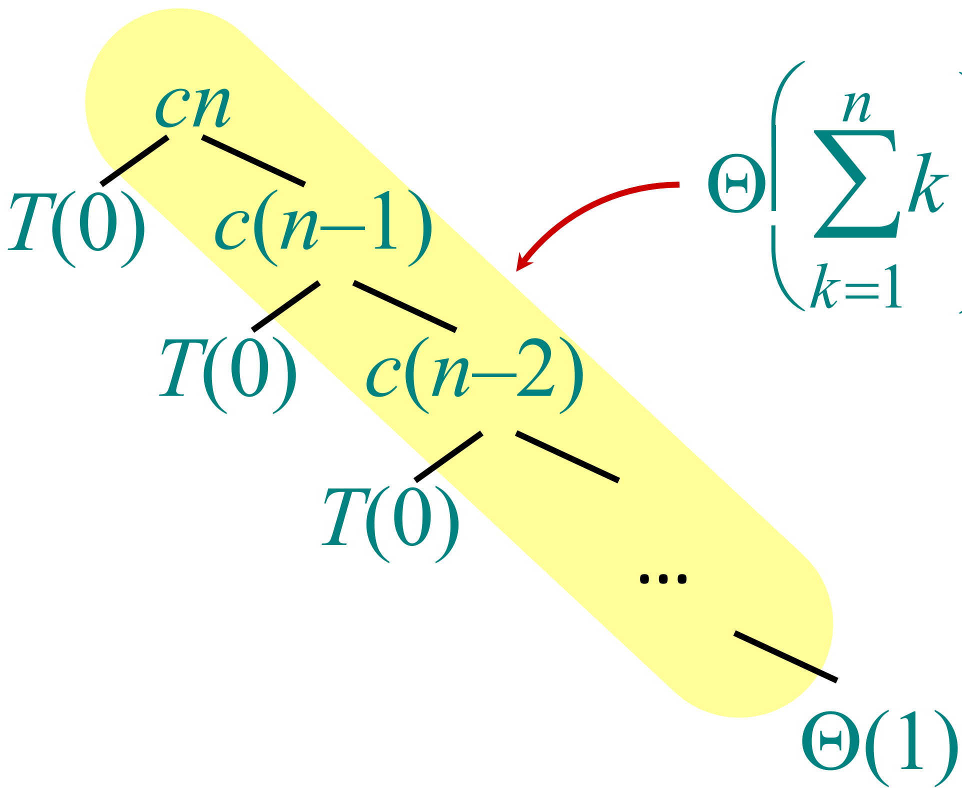
$$T(n) = T(0) + T(n-1) + cn$$

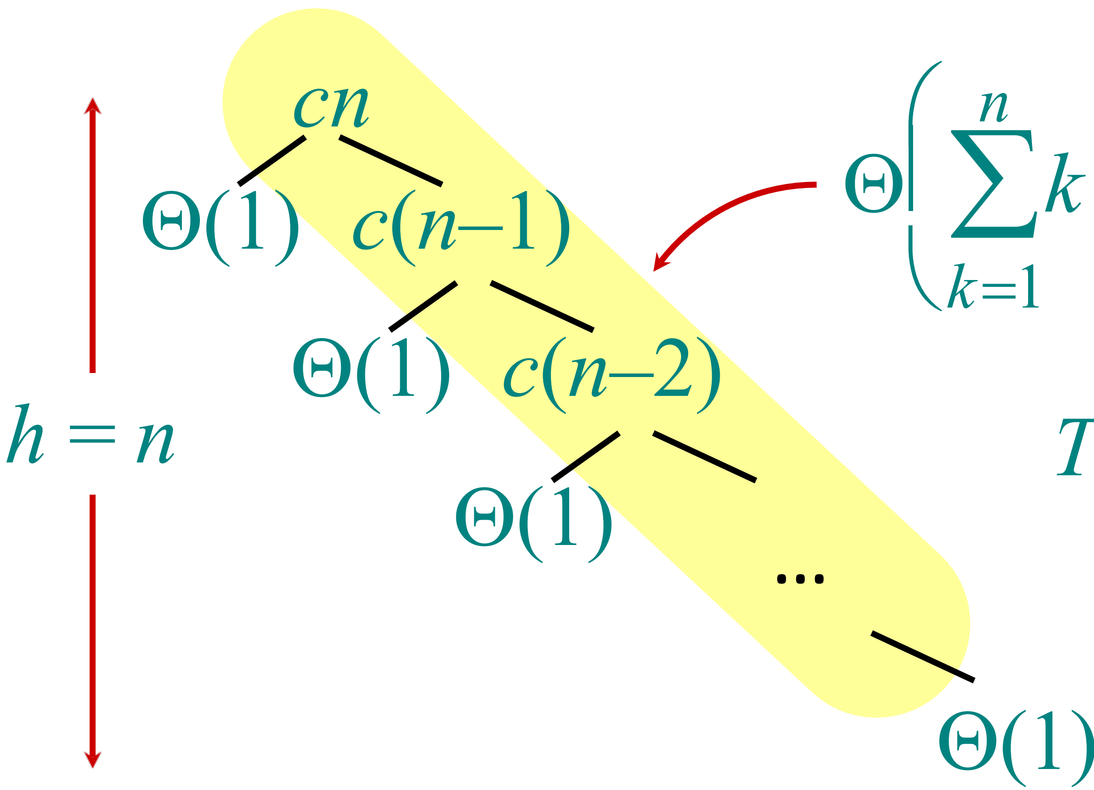




$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

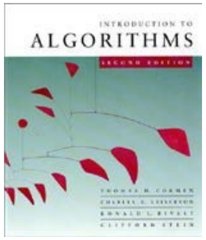
# Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$




$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$




# Best-case analysis

***(For intuition only!)***

If we're lucky, PARTITION splits the array evenly:

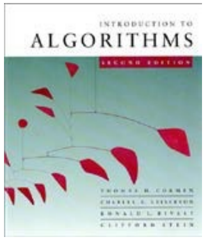
$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

(same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$ ?

$$T(n) = T\left(\tfrac{1}{10} n\right) + T\left(\tfrac{9}{10} n\right) + \Theta(n)$$

What is the solution to this recurrence?



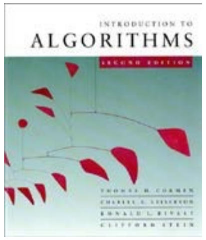

# Analysis of “almost-best” case

$$T(n)$$

# Analysis of “almost-best” case

$$cn$$

$$T\left(\tfrac{1}{10}n\right) \qquad\qquad T\left(\tfrac{9}{10}n\right)$$

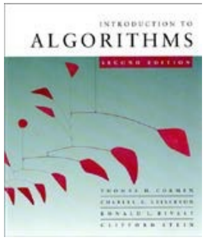



# Analysis of “almost-best” case

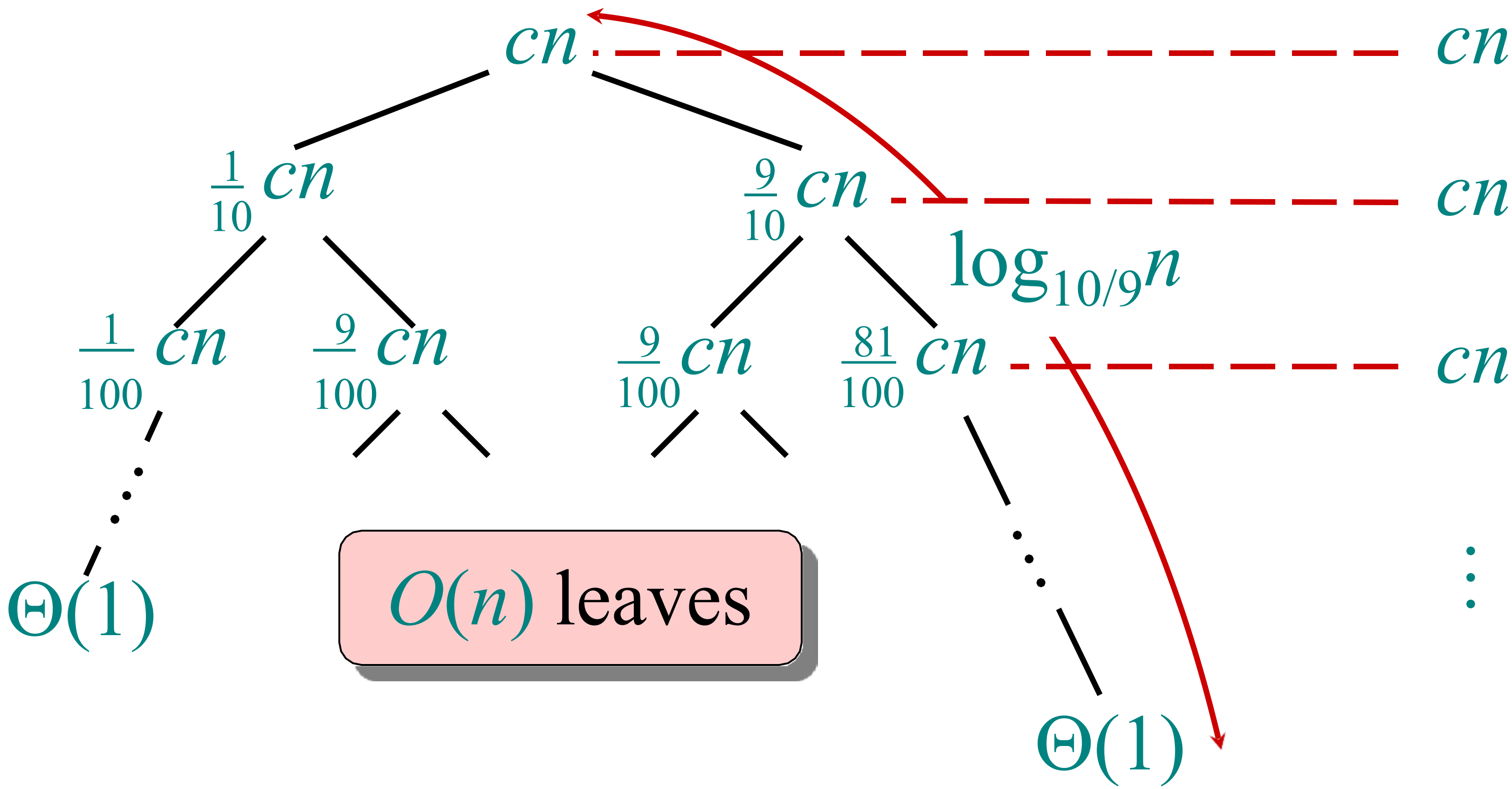
$$cn$$

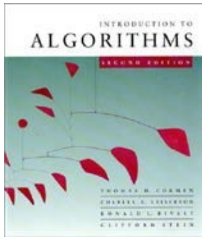
$$\frac{1}{10}cn \qquad \frac{9}{10}cn$$

$$T\left(\frac{1}{100}n\right) \; T\left(\frac{9}{100}n\right) \qquad T\left(\frac{9}{100}n\right) \; T\left(\frac{81}{100}n\right)$$

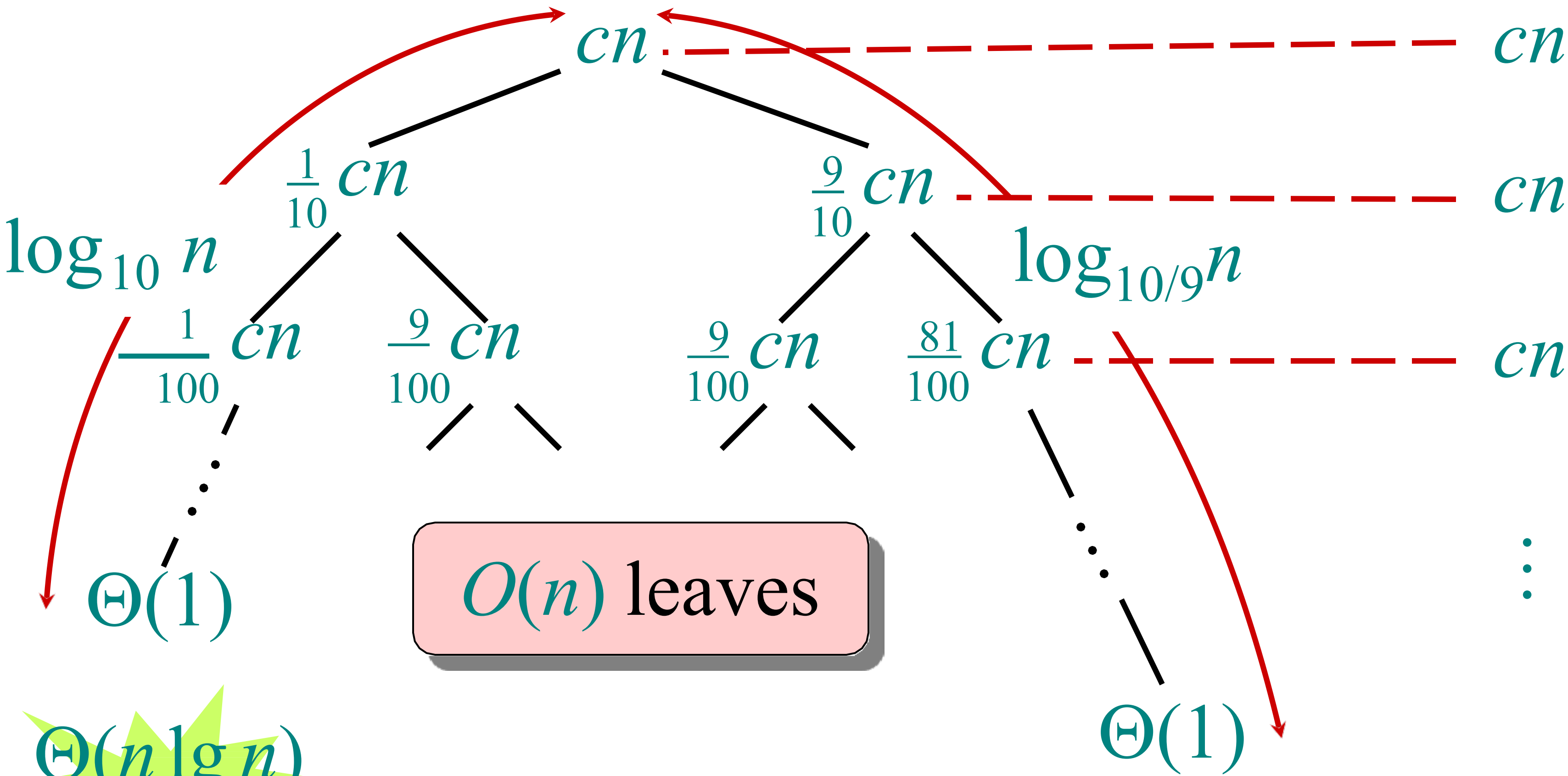


# Analysis of “almost-best” case






# Analysis of "almost-best" case



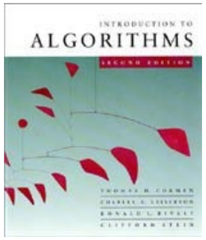

$\Theta(n \lg n)$

***Lucky!***

$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

*Only counting levels according to the shallowest root-to-leaf path*

Why ?




# More intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

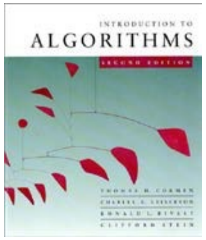
$$L(n) = 2U(n/2) + \Theta(n) \quad \textit{lucky}$$
$$U(n) = L(n-1) + \Theta(n) \quad \textit{unlucky}$$

Solving:

$$\begin{aligned} L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2-1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

***Lucky!***

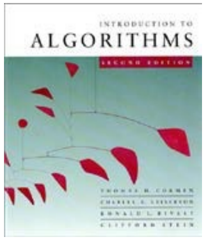
How can we make sure we are usually lucky?




# Randomized quicksort

**IDEA**: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

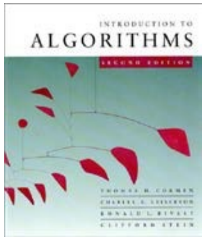



# Randomized quicksort analysis

- Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size $n$, assuming random numbers are independent.
- For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

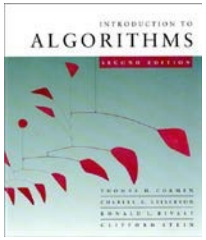
$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.




# Analysis (continued)

$$
T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad \ldots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}
$$

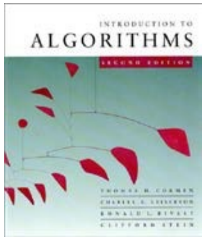
$$
= \sum_{k=0}^{n-1} X_k \big( T(k) + T(n-k-1) + \Theta(n) \big)
$$




# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

Take expectations of both sides.

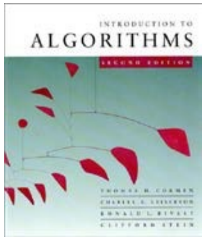



# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Linearity of expectation.



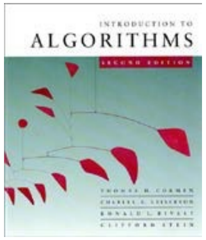

# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)]$$

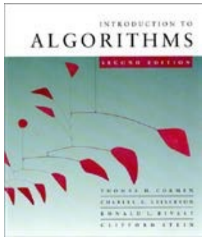
Independence of $X_k$ from other random choices.




# Calculating expectation

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)
\end{aligned}
$$

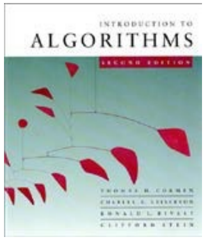
Linearity of expectation; $E[X_k] = 1/n$ .




# Calculating expectation

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E\left[X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n) \\
&= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)
\end{aligned}
$$

Summations have identical terms.




# Hairy recurrence

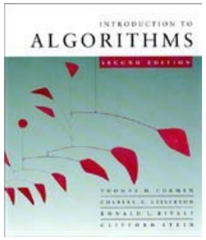
$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

**Prove:** $E[T(n)] \le an \lg n$ for constant $a > 0$.

- Choose $a$ large enough so that $an \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

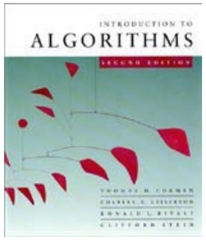
**Use fact:** $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).




# Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.

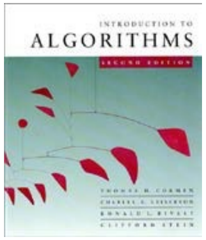



# Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.



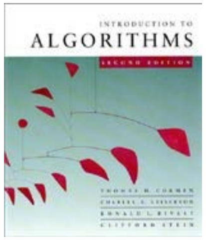

# Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired* – *residual***.




# Substitution method

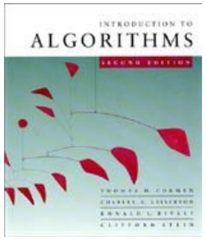
$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$= \frac{2a}{n}\left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2\right) + \Theta(n)$$

$$= an \lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

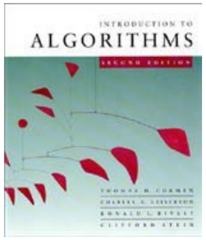
$$\le an \lg n \, ,$$

if $a$ is chosen large enough so that $an/4$ dominates the $\Theta(n)$.




# Understand the Result

- Expected cost of randomized qsort is O($n$ lg $n$)
  - For the same data instance, run rand-qsort multiple times
  - Take the average running time, and it is upper bounded by O(n lg n)




# Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.