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Dynamic Programming (I)

- Fibonacci Numbers
- Matrix chain multiplication
- Knapsack Problem
- RNA secondary structure

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Fibonacci Numbers

- Definition

$$f(n) = \begin{cases} 0 & \text{if } \; n = 0 \\ 1 & \text{if } \; n = 1 \\ F(n-1) + F(n-2) & \text{if } \; n > 1 \end{cases}$$

- The first several numbers are:
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 , 144 …
- Question: Given n, how to compute F(n)?
  - Recursion

Leonardo Fibonacci

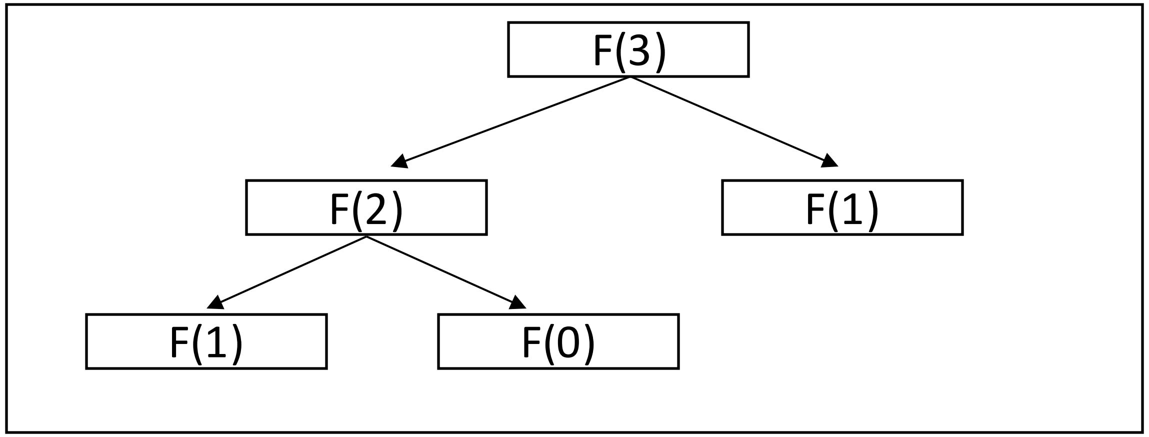
Fibonacci Numbers – Naïve Algorithm

- Computing the n<u>th</u> Fibonacci number recursively:

$$f(n)=\begin{cases} 0 & if \quad n=0 \\ 1 & if \quad n=1 \\ F(n-1)+F(n-2) & if \quad n>1 \end{cases}$$

code ...

```
def Fib(n):
   if (n <= 1)
      return n;
   else
      return Fib(n - 1) + Fib(n - 2);
```



Fibonacci Numbers – Naïve Algorithm

- Running time

$$T(n) = T(n-1) + T(n-2) + O(1)$$

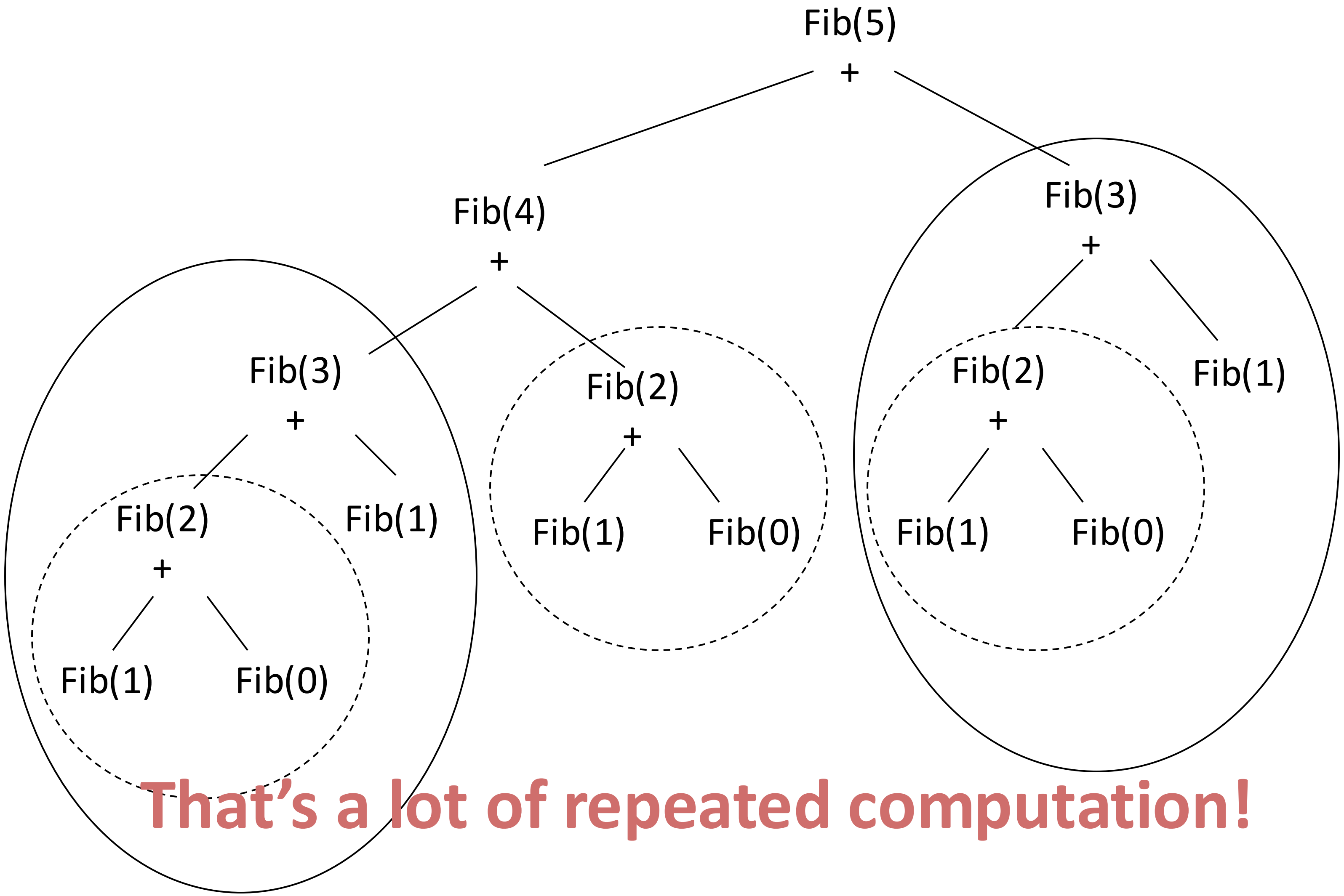
$$\Rightarrow T(n) \geq T(n-1) + T(n-2) \text{ for } n \geq 2$$

$$\Rightarrow T(n) \geq 2T(n-2)$$

- What is the solution to this?
  - Clearly it is $O(2^n)$, but this is not tight.
  - A lower bound is $\Omega(2^{n/2})$.
  - You should notice that $T(n)$ grows as fast as the Fibonacci numbers $F(n)$, so in fact $T(n) = \Theta(F(n))$.

Fibonacci Numbers – Naïve Algorithm

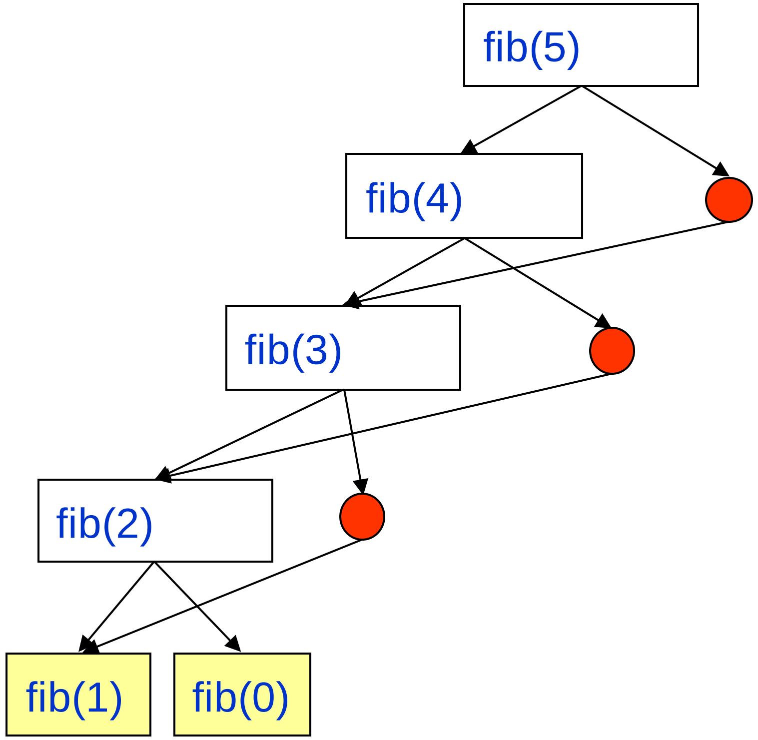
- What's going on with this naïve approach?



That's a lot of repeated computation!

Memoization

- Memoization frees us from redundant calculations ☺
  - Remember solutions of all the sub-problems
  - Trade space for time



Sub-problem	Opt Solution
fib(0)	0
fib(1)	1
fib(2)	1
fib(3)	2
fib(4)	3

Fibonacci Numbers

– Computing the nth Fibonacci number using as follow:
  – F(0) = 0
  – F(1) = 1
  – F(2) = 1+0 = 1
  – …
  – F(n-2) =
  – F(n-1) =
  – F(n) = F(n-1) + F(n-2)

code …

```
def fasterFibonacci(n):
    F = [0, 1, None, None, …, None ]
        \\ F has length n + 1
    for i = 2, …, n:
        F[i] = F[i-1] + F[i-2]
    return F[n]
```

0	**1**	**1**	**. . .**	***F(n-2)***	***F(n-1)***	***F(n)***

- Efficiency:
  – Time – O(n)
  – Space – O(n) ➔ can be improved to O(1)

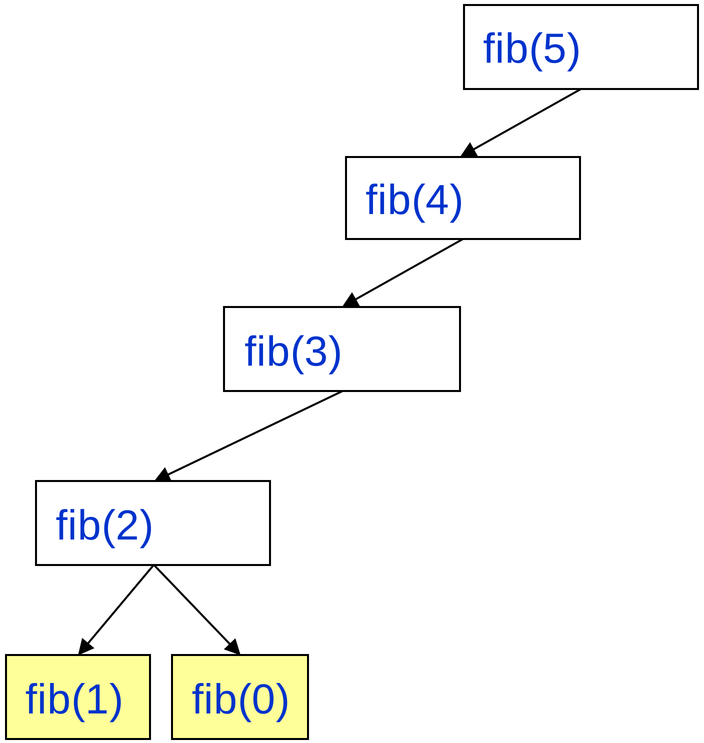
This is an example of dynamic programming ☺

Dynamic Programming

- Ideas
  - Ensure all needed recursive calls are already computed and memorized

    ➔ a good schedule of computation
  - (Optional) Reused space to store previous recursive call results

  ➔ Arrive at the same efficient (special) solution for Fib()



“ Those who cannot remember the past are condemned to repeat it. ”

— Dynamic Programming

Dynamic Programming

- Dynamic Programming is an algorithm design technique for *optimization problems:* often minimizing or maximizing.

- Like divide and conquer, DP solves problems by combining solutions to sub-problems.

- Unlike divide and conquer, sub-problems are not independent.
  - DP breaks up a problem into a series of overlapping sub-problems.
    - i.e, Both F[i+1] and F[i+2] directly use F[i]. And lots of different F[i+x] indirectly use F[i].

Main Ideas

1. Recursion: Divide the problem into sub-problems, so that their solutions can be combined into a solution to the problem.

2. Tabulation of sub-problems: Solve each sub-problem just once and save its solution in a “look-up” table.

Dynamic Programming

- The term Dynamic Programming comes from Control Theory, not computer science. Programming refers to the use of tables (arrays) to construct a solution.

- In Dynamic Programming, we usually reduce time by increasing the amount of space.

- We solve the problem by solving sub-problems of increasing size and saving each optimal solution in a table (usually).

- The table is then used for finding the optimal solution to larger problems.

- Time is saved since each sub-problem is solved only once.

Two Ways to Think and Implement DP

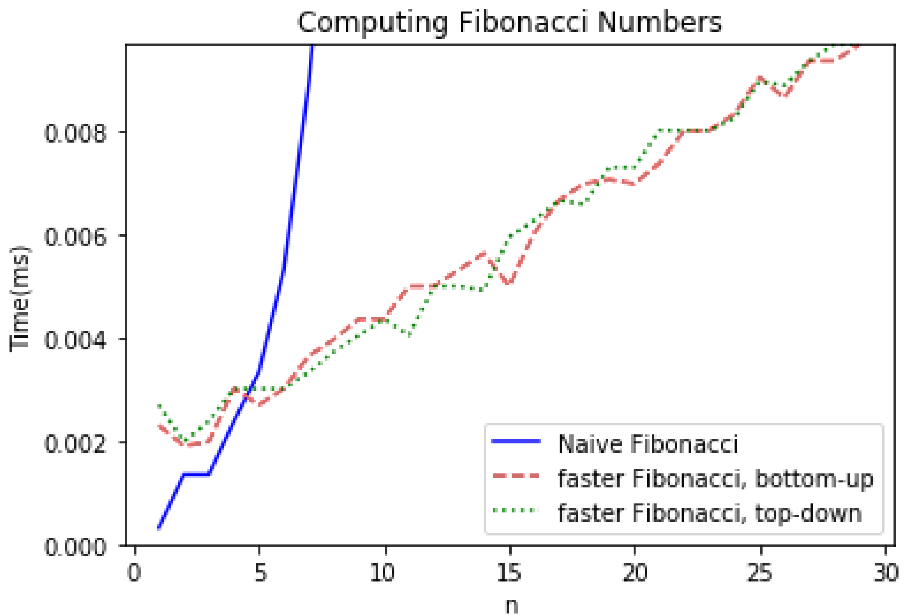
- Top down:
- Think of it like a recursive algorithm.
- To solve the big problem:
  - Recurse to solve smaller problems
    - Those recurse to solve smaller problems
      - etc..
- The difference from divide and conquer:
  - Keep track of what small problems you've already solved to prevent re-solving the same problem twice.
  - Aka, "**memoization**"

- Bottom up:
- For Fibonacci:
- Solve the small problems first
  - fill in F[0],F[1]
- Then bigger problems
- …
- Then bigger problems
  - fill in F[n-1]
- Then finally solve the real problem.
  - fill in F[n]

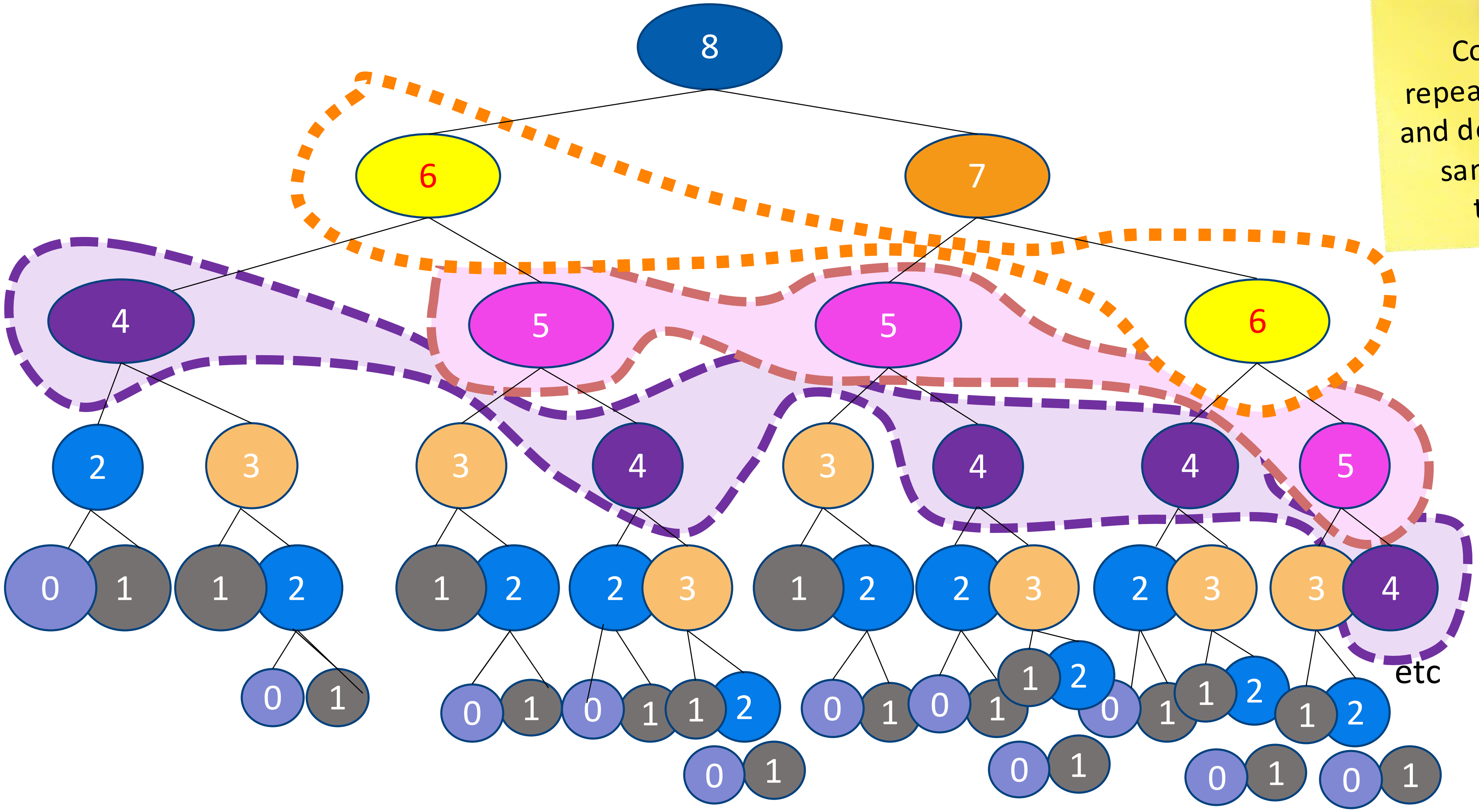
Example of Top-Down Fibonacci

```
define a global list F = [0,1,None, None, …, None]
def Fibonacci(n):
    if F[n] != None:
        return F[n]
    else:
        F[n] = Fibonacci(n-1) + Fibonacci(n-2)
    return F[n]
```

Memoization: Keeps track (in F) of the stuff you've already done.



Memoization Visualization



Collapse repeated nodes and don't do the same work twice!

Dynamic Programming

- Underpins many optimization problems, e.g.,
  - Matrix Chaining optimization
  - Longest Common Subsequence
  - 0-1 Knapsack Problem
  - Shortest path

- Next we will give many example problems to help understand the basic idea of Dynamic Programming.

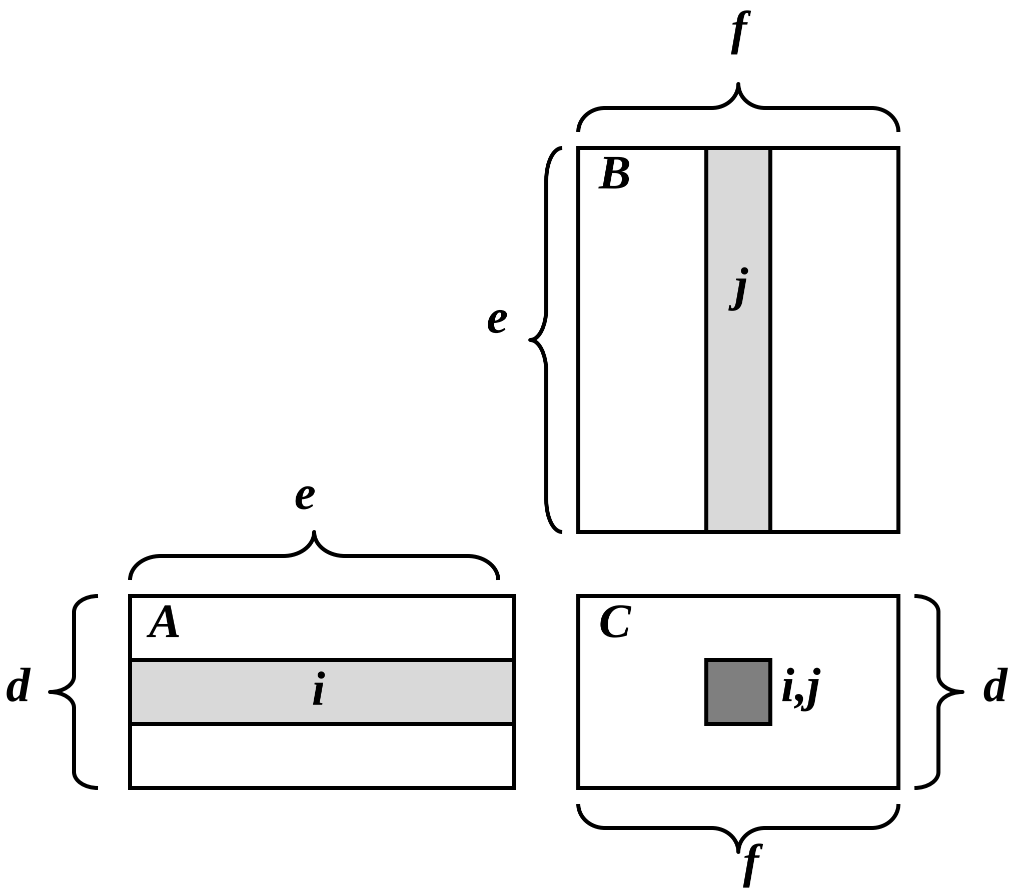
Recipe for Applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up.

Matrix Chain Multiplication

- Review: Matrix Multiplication.
  - $\boldsymbol{C} = \boldsymbol{A} * \boldsymbol{B}$
  - $\boldsymbol{A}$ is $\boldsymbol{d} \times \boldsymbol{e}$ and $\boldsymbol{B}$ is $\boldsymbol{e} \times \boldsymbol{f}$
  - $\boldsymbol{O(d \cdot e \cdot f)}$ time

$$C[i,j] = \sum_{k=0}^{e-1} A[i,k] * B[k,j]$$



Matrix Chain Multiplication

- **Matrix Chain Multiplication:**
  - Compute $A=A_0*A_1*\ldots*A_{n-1}$
  - A_i is $d_i \times d_{i+1}$
  - Problem: How to parenthesize?
- Example
  - B is 3×100
  - C is 100×5
  - D is 5×5
  - (B*C)*D takes 1500 + 75 = 1575 ops
    - (3 x 100 x 5) + (3 x 5 x 5)
  - B*(C*D) takes 1500 + 2500 = 4000 ops

Enumeration Approach for MCM

- **Matrix Chain Multiplication Alg.:**
  - Try all possible ways to parenthesize $A=A_0*A_1*\ldots*A_{n-1}$
  - Calculate number of ops for each one
  - Pick the one that is best
- Running time:
  - The number of parenthesizations is equal to the number of binary trees with $n-1$ nodes
  - This is **exponential**!
  - It is called the Catalan number, and it is almost 4^n.
  - This is a terrible algorithm!

Greedy Approach for MCM

- Idea #1: repeatedly select the product that uses the fewest operations.
- Counter-example:
  - A is 101 $\times$ 11
  - B is 11 $\times$ 9
  - C is 9 $\times$ 100
  - D is 100 $\times$ 99
  - Greedy idea #1 gives A*((B*C)*D)), which takes 109989+9900+108900=228789 ops
  - (A*B)*(C*D) takes 9999+89991+89100=189090 ops
- The greedy approach is not giving us the optimal value.

Dynamic Programming Approach for MCM

- The optimal solution can be defined in terms of optimal sub-problems
  - There has to be a final multiplication (root of the expression tree) for the optimal solution.
  - Say, the final multiplication is at index k: $(A_0 * \ldots * A_k) * (A_{k+1} * \ldots * A_{n-1})$.
- Let us consider all possible places for that final multiplication:
  - There are n-1 possible ***splits***. Assume we know the minimum cost of computing the matrix product of each combination $A_0 \ldots A_i$ and $A_{i+1} \ldots A_{n-1}$. Let's call these $N_{0,i}$ and $N_{i+1,n-1}$.
- Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix, and the final result will be a $d_0 \times d_n$.

Dynamic Programming Approach for MCM

- Define the following:

$$\boxed{N_{0,n-1} = \min_{0 \le k < n-1} \{N_{0,k} + N_{k+1,n-1} + d_0 d_{k+1} d_n\}}$$

- Then the optimal solution $N_{0,n-1}$ is the sum of two optimal sub-problems, $N_{0,k}$ and $N_{k+1,n-1}$ plus the time for the last multiplication.

Dynamic Programming Approach for MCM

- Define **sub-problems**:
  - Find the best parenthesization of an arbitrary set of consecutive products: $A_i * A_{i+1} * \ldots * A_j$.
  - Let $N_{i,j}$ denote the **minimum** number of operations done by this sub-problem.
    - Define $N_{k,k} = 0$ for all k.
  - The optimal solution for the whole problem is then $N_{0,n-1}$.

Dynamic Programming Approach for MCM

- The characterizing equation for $N_{i,j}$ is:

$$N_{i,j} = \min_{i \leq k < j} \{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

- Note that, for example $N_{2,6}$ and $N_{3,7}$, both need solutions to $N_{3,6}$, $N_{4,6}$, $N_{5,6}$, and $N_{6,6}$. Solutions from the set of no matrix multiplies to four matrix multiplies.
  - This is an example of high sub-problem overlap, and clearly pre-computing these will significantly speed up the algorithm.

Recursive Approach

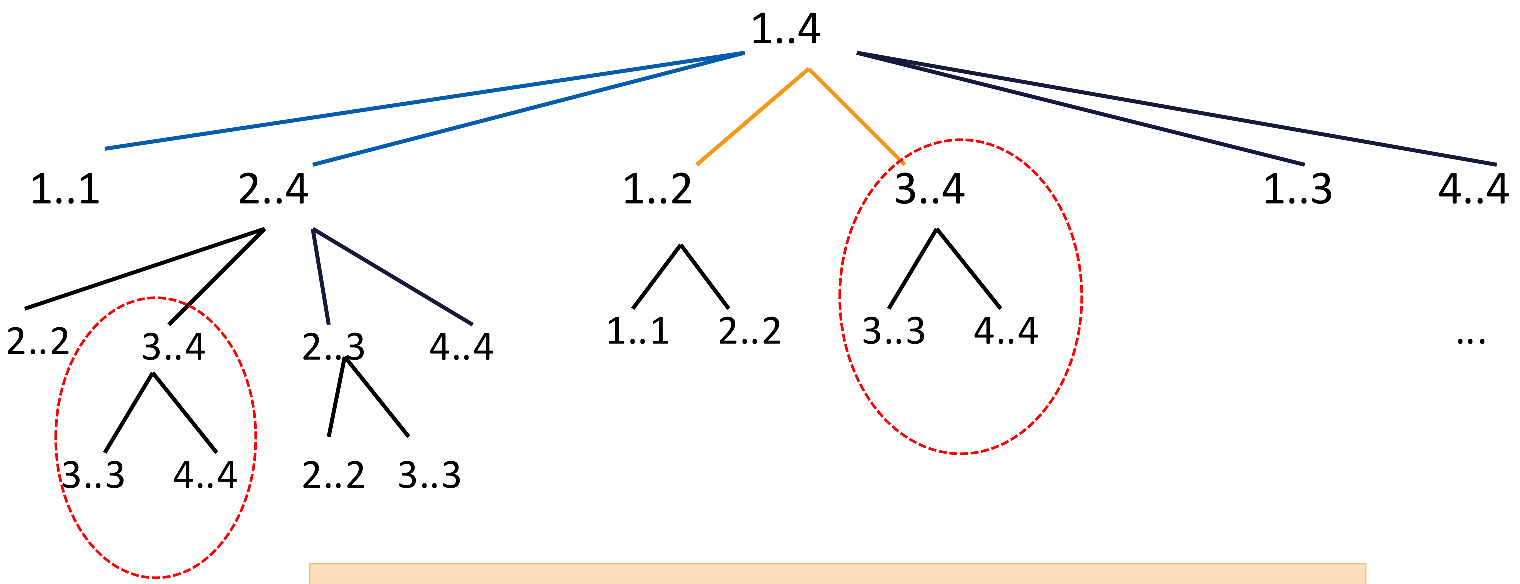
- We could implement the calculation of these $N_{i,j}$'s using a straightforward recursive implementation of the equation (aka not pre-compute them).

Algorithm ***RecursiveMatrixChain***(***S, i, j***):

    Input: sequence ***S*** of ***n*** matrices to be multiplied
    Output: number of operations in an optimal parenthesization of ***S***
    if i=j
        then return 0
    for ***k*** $\leftarrow$ i **to** ***j*** **do**
        $N_{i,j} \leftarrow \min\{N_{i,j},$ ***RecursiveMatrixChain***(***S, i ,k***)
            $+$ ***RecursiveMatrixChain***(***S, k+1,j***) $+ d_i d_{k+1} d_{j+1}\}$
    return $N_{i,j}$

Subproblem Overlap

$$N_{i,j} = \min_{i \le k < j} \{N_{i,k} + N_{k+1,j} + \ldots\}$$



How to schedule the sub-problems?

Dynamic Programming Algorithm

- High sub-problem overlap, with independent sub-problems indicate that a dynamic programming approach may work.
- Construct optimal sub-problems “bottom-up.” and remember them.
- $N_{i,i}$’s are easy, so start with them
- Then do problems of *length* 2,3,… sub-problems, and so on.
- Running time: $O(n^3)$

Dynamic Programming Algorithm

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Algorithm *matrixChain*(*S*):

Input: sequence ***S*** of ***n*** matrices to be multiplied

Output: number of operations in an optimal parenthesization of ***S***

for $i \leftarrow 1$ **to** $n - 1$ **do**

$\quad N_{i,i} \leftarrow 0$

for $b \leftarrow 1$ **to** $n - 1$ **do**

$\quad$ { $b = j - i$ is the length of the problem }

$\quad$ **for** $i \leftarrow 0$ **to** $n - b - 1$ **do**

$\quad\quad j \leftarrow i + b$

$\quad\quad N_{i,j} \leftarrow +\infty$

$\quad\quad$ **for** $k \leftarrow i$ **to** $j - 1$ **do**

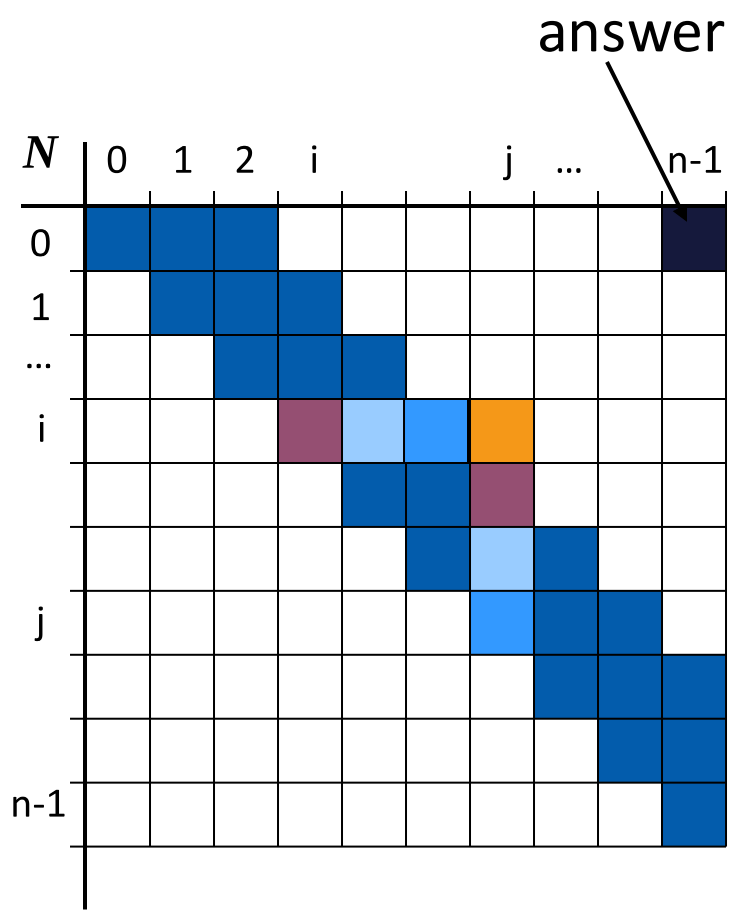
$\quad\quad\quad N_{i,j} \leftarrow \min\{N_{i,j}, N_{i,k} + N_{k+1,j} + d_i\, d_{k+1}\, d_{j+1}\}$

return $N_{0,n-1}$

Algorithm Visualization

- The bottom-up construction fills in the N array by diagonals
- $N_{i,j}$ gets values from previous entries in i-th row and j-th column
- Filling in each entry in the N table takes O(n) time.
- Total run time: $O(n^3)$
- Getting actual parenthesization can be done by remembering “k” for each N entry

$$N_{i,j} = \min_{i \le k < j} \{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$



Algorithm Visualization

- A_0: 30 X 35; A_1: 35 X15; A_2: 15X5;
  A_3: 5X10; A_4: 10X20; A_5: 20 X 25

0	1	2	3	4	5	
0	15,750	7,875	9,375	11,875	15,125	0
	0	2,625	4,375	7,125	10,500	1
		0	750	2,500	5,375	2
			0	1,000	3,500	3
				0	5,000	4
					0	5

$$N_{i,j} = \min_{i \le k < j} \{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

$$\begin{aligned}
&N_{1,4} = \min\{ \\
&N_{1,1} + N_{2,4} + d_1 d_2 d_5 = 0 + 2500 + 35 * 15 * 20 = 13000, \\
&N_{1,2} + N_{3,4} + d_1 d_3 d_5 = 2625 + 1000 + 35 * 5 * 20 = 7125, \\
&N_{1,3} + N_{4,4} + d_1 d_4 d_5 = 4375 + 0 + 35 * 10 * 20 = 11375 \\
&\} \\
&= 7125
\end{aligned}$$

Algorithm Visualization

$$(A_0*(A_1*A_2))*((A_3*A_4)*A_5)$$

0	1	2	3	4	5	
0	15,750	7,875	9,375	11,875	15,125	0
	0	2,625	4,375	7,125	10,500	1
		0	750	2,500	5,375	2
			0	1,000	3,500	3
				0	5,000	4
					0	5

0	1	2	3	4	5	
	0	0	2	2	2	0
		1	2	2	2	1
			2	2	2	2
				3	4	3
					4	4
						5

Matrix Chain Multiplication

- Some final thoughts
  - We ~~reduced~~ replaced a $\boldsymbol{O}(2^n)$ algorithm with a $\Theta(n^3)$ algorithm.
  - While the generic top-down recursive algorithm would have solved $\boldsymbol{O}(2^n)$ sub-problems, there are $\Theta(n^2)$ sub-problems.
    - Implies a high overlap of sub-problems.
  - The sub-problems are independent:
    - Solution to $A_0A_1 \ldots A_k$ is independent of the solution to $A_{k+1} \ldots A_n$.

Matrix Chain Multiplication Summary

- Determine the cost of each pair-wise multiplication, then the ***minimum*** cost of multiplying three consecutive matrices (2 possible choices), using the pre-computed costs for two matrices.
- Repeat until we compute the minimum cost of all n matrices using the costs of the minimum n-1 matrix product costs.
  - n-1 possible choices.

The 0/1 Knapsack Problem

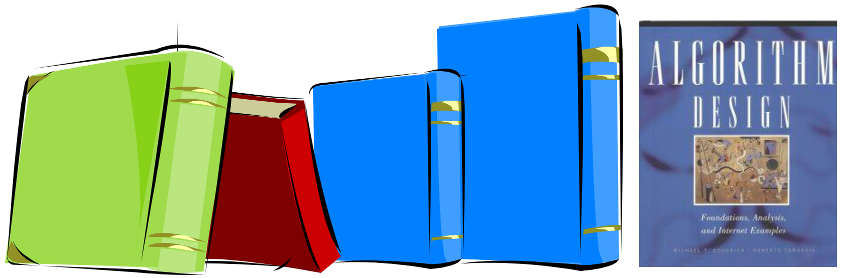
- Given: A set S of *n* items (one piece each), with each item *i* having
  - w_i - a positive weight
  - b_i - a positive benefit
- Goal: Choose items with maximum total benefit but with weight at most W.
- If we are **not** allowed to take fractional amounts, then this is the **0/1 knapsack problem**.
  - In this case, we let T denote the set of items we take
  - Objective: maximize $\sum_{i \in T} b_i$
  - Constraint: $\sum_{i \in T} w_i \leq W$

Linear Programming formulation

Example

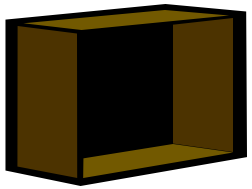
- Given: A set S of n items, with each item i having
  - b_i - a positive “benefit”
  - w_i - a positive “weight”
- Goal: Choose items with maximum total benefit but with weight at most W.

Items:



	1	2	3	4	5
Weight:	4 in	2 in	2 in	6 in	2 in
Benefit:	$20	$3	$6	$25	$80

“knapsack”



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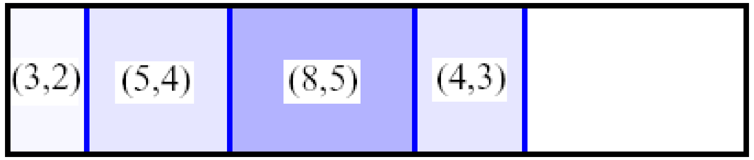
Solution:

- item 5 ($80, 2 in)
- item 3 ($6, 2 in)
- item 1 ($20, 4 in)

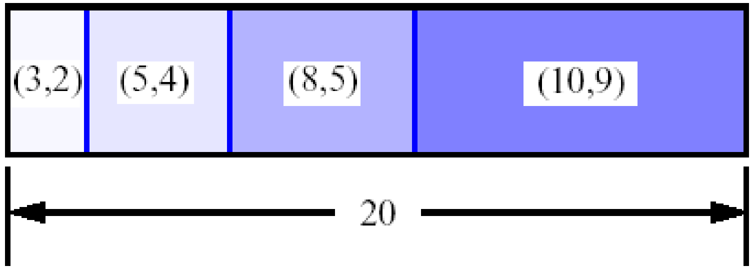
First Attempt

- S_k: Set of items numbered 1 to k.
- Define B[k] = best selection from S_k.
- Problem: does not have sub-problem optimality:
  - Consider set S={(3,2),(5,4),(8,5),(4,3),(10,9)} of (benefit, weight) pairs and total weight W = 20

Best for S_4:



Best for S_5:



Second Attempt

- S_k: Set of items numbered 1 to k.
- Define B[k,w] to be the best selection from S_k with weight at most *w*
- This does have sub-problem optimality.

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- I.e., the best subset of S_k with weight at most *w* is either:
  - the best subset of S_{k-1} with weight at most *w* or
  - the best subset of S_{k-1} with weight at most $w-w_k$ plus item *k*

Knapsack Example

Knapsack of capacity $W = 5$

$w_1 = 2, v_1 = 12$ $w_2 = 1, v_2 = 10$

$w_3 = 3, v_3 = 20$ $w_4 = 2, v_4 = 15$

item	weight	value
1	2	\$12
2	1	\$10
3	3	\$20
4	2	\$15

Max item allowed	Max Weight					
	0	**1**	**2**	**3**	**4**	**5**
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12	22	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	**37**

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k] + b_k\} & \text{else} \end{cases}$$

Algorithm

- Since B[k,w] is defined in terms of B[k–1,*], we can use two arrays of instead of a matrix.
- Running time is **O**(nW).
- Not a polynomial-time algorithm since W may be large.
- Called a pseudo-polynomial time algorithm.

Algorithm

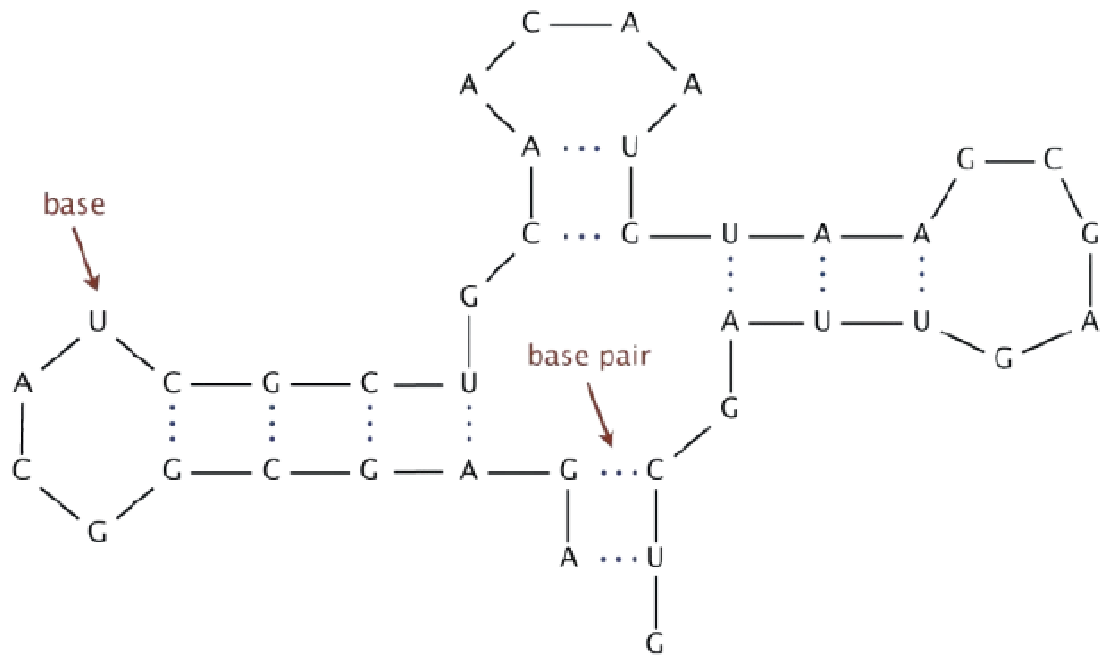
Input: set S of n items with benefit b_i and weight w_i; maximum weight W

Output: benefit of best subset of S with weight at most W

```
let A and B be arrays of length W + 1
for w ← 0 to W do
    B[w] ← 0
for k ← 1 to n do
    copy array B into array A
    for w ← w_k to W do
        if A[w−w_k] + b_k > A[w] then
            B[w] ← A[w−w_k] + b_k
return B[W]
```

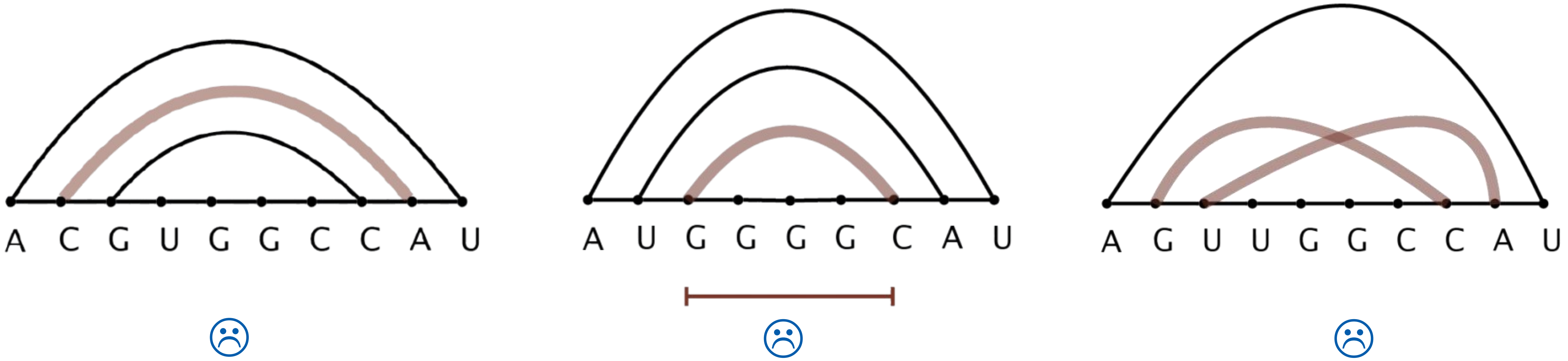
RNA secondary structure

- RNA: String B = b1b2…bn over alphabet { A, C, G, U }.
  - e.g. GUCGAUUGAGCGAAUGUAACAACGUGGCUACGGCGAGA
- Secondary structure: RNA is single-stranded so it tends to loop back and form base pairs with itself. This structure is essential for understanding behavior of molecule.



RNA secondary structure prediction

- For a given RNA sequences B, Finding a set of pairs $S=\{(b_i,b_j)\}$ that satisfy:
  - [Watson–Crick complement] $(b_i,b_j) \in \{A–U, U–A, C–G, G–C\}$.
  - [No sharp turns] If (b_i,b_j), then $i < j - 4$.
  - [Non-crossing] If (b_i,b_j) and (b_k, b_l) are two pairs in S, then we cannot have $i < k < j < l$.



RNA secondary structure prediction

- RNA tends to form the secondary structure with more base pairs.