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Greedy Algorithms

- Activity selection
- Activity selection version 2
- Minimum Spanning Trees

Yanlin Zhang & Wei Wang | DSAA 2043 Spring 2025

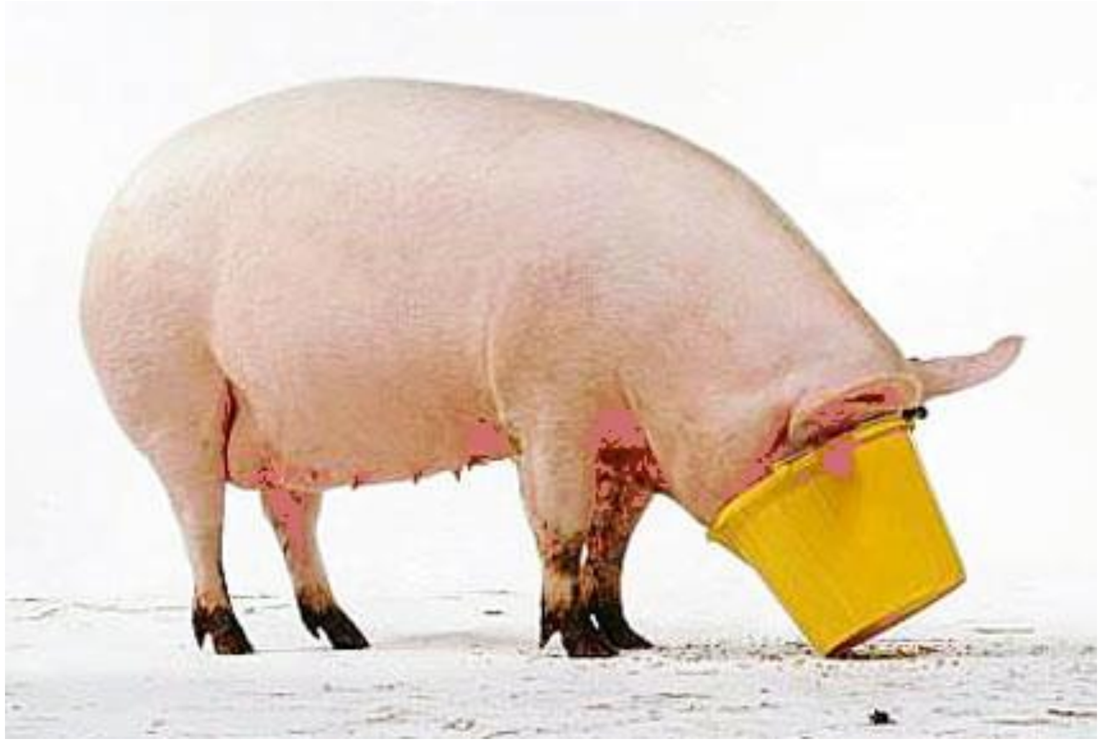
Greedy Algorithms

- Make choices one-at-a-time.

(grow) partial solutions

- Never look back.

- Hope for/prove the best.



Today

One example of a greedy algorithm that **does not work**:

Knapsack again

Three examples of greedy algorithms that **do work**:

Activity Selection

Job Scheduling

Minimum Spanning Tree

Non-example: Unbounded Knapsack

Capacity: 10

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

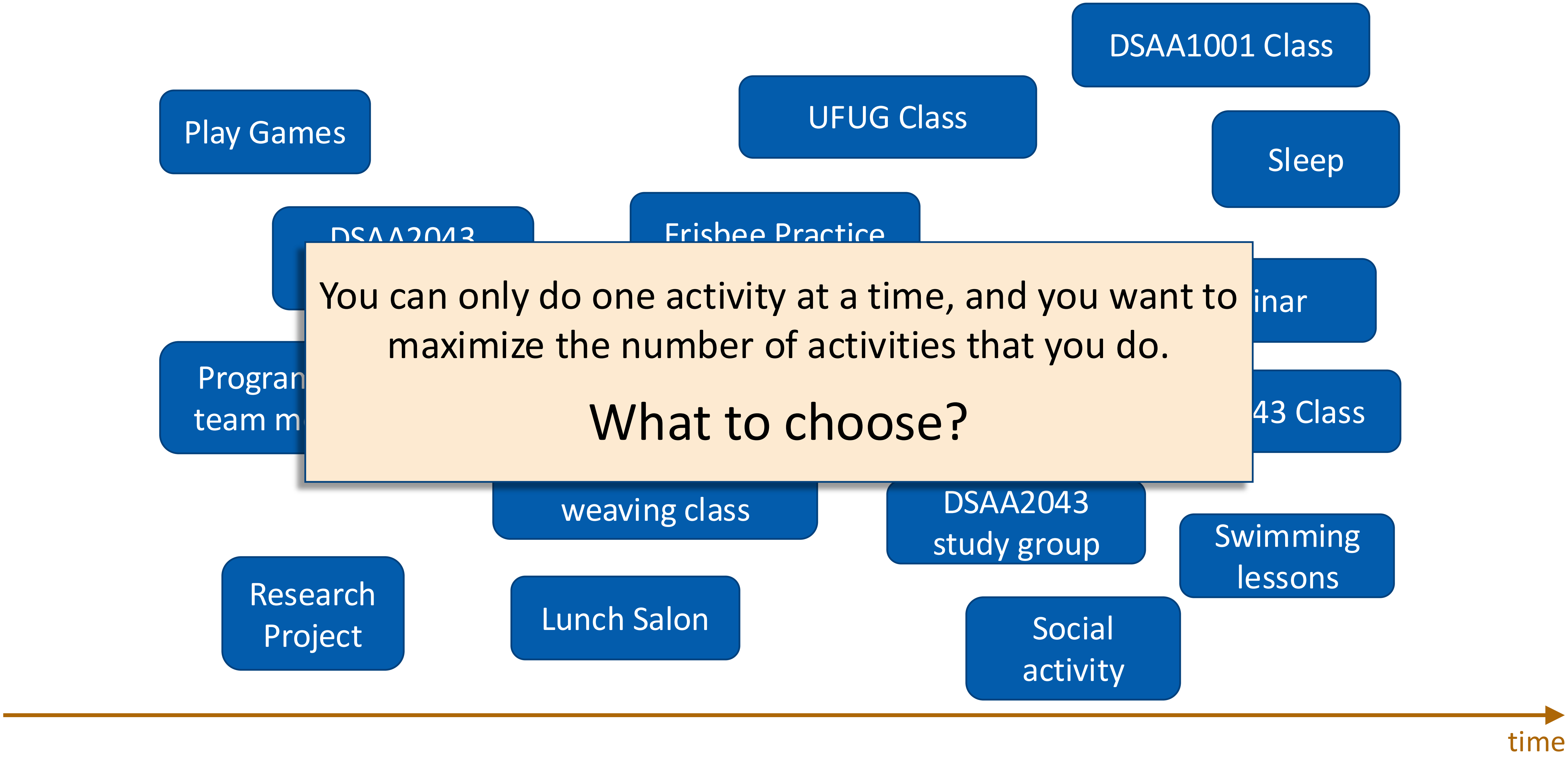
- Unbounded Knapsack:
  - Suppose I have infinite copies of all items.
  - What's the most valuable way to fill the knapsack?

Total weight: 10
Total value: 42

- "Greedy" algorithm for unbounded knapsack:
  - Tacos have the best Value/Weight ratio!
  - Keep grabbing tacos!

Total weight: 9
Total value: 39

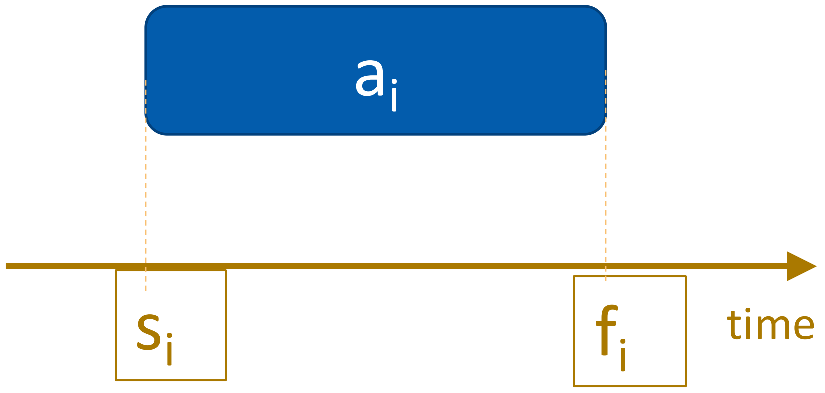
Example where greedy works



You can only do one activity at a time, and you want to maximize the number of activities that you do.

What to choose?

Activity selection

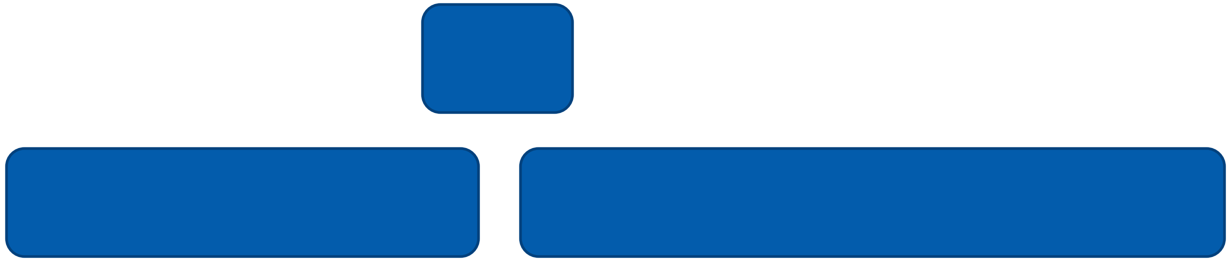


- Input:
  - Activities $a_1, a_2, \ldots, a_n$
  - Start times $s_1, s_2, \ldots, s_n$
  - Finish times $f_1, f_2, \ldots, f_n$
- Output:
  - A way to maximize the number of activities you can do today.

In what order should you greedily add activities?

In what order?

- Shortest job first?

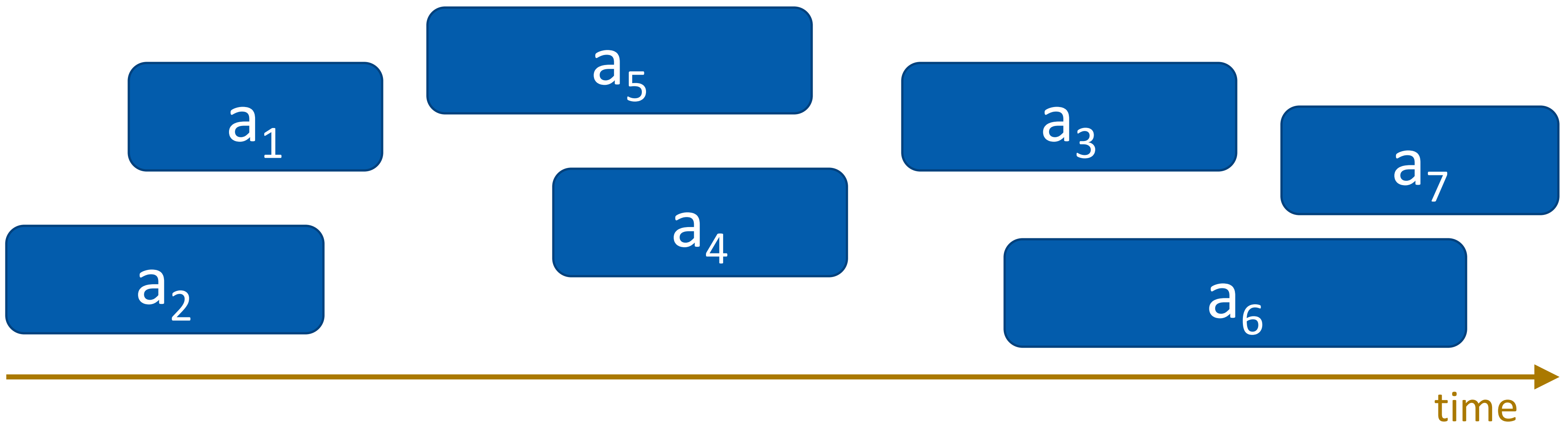


- Earliest start time?

- Earliest finish time? ✓

Greedy Algorithm

- Pick activity you can add with the smallest finish time.
- Repeat.



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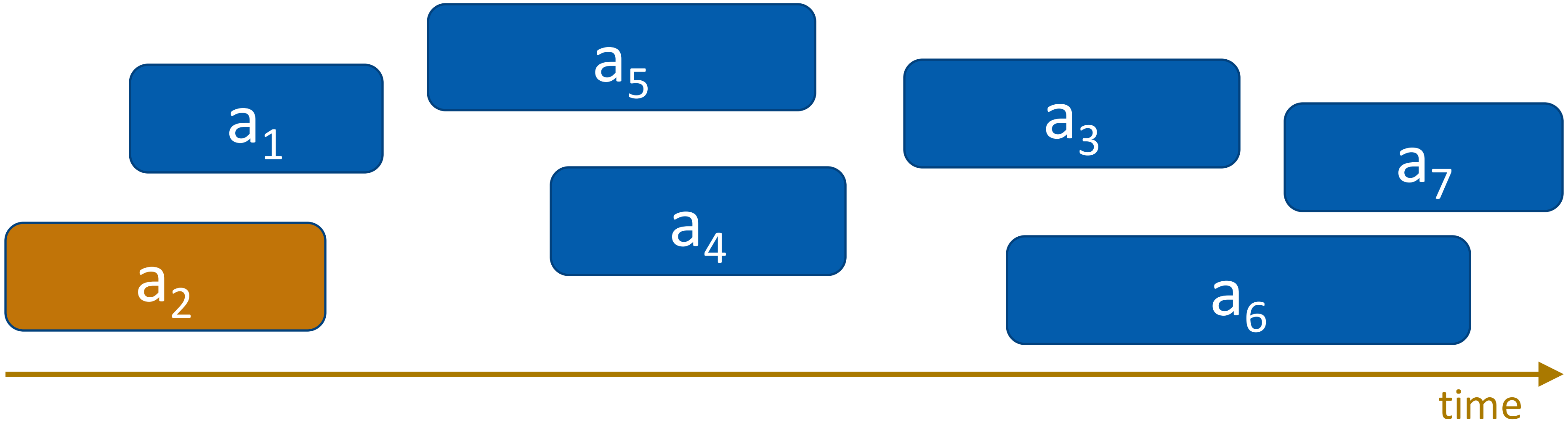

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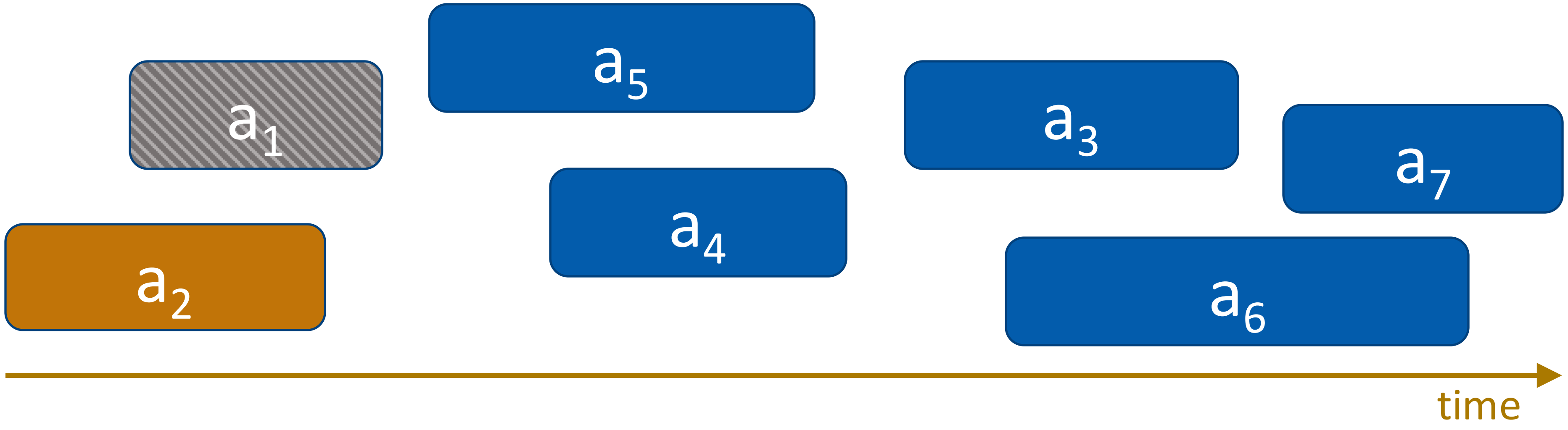

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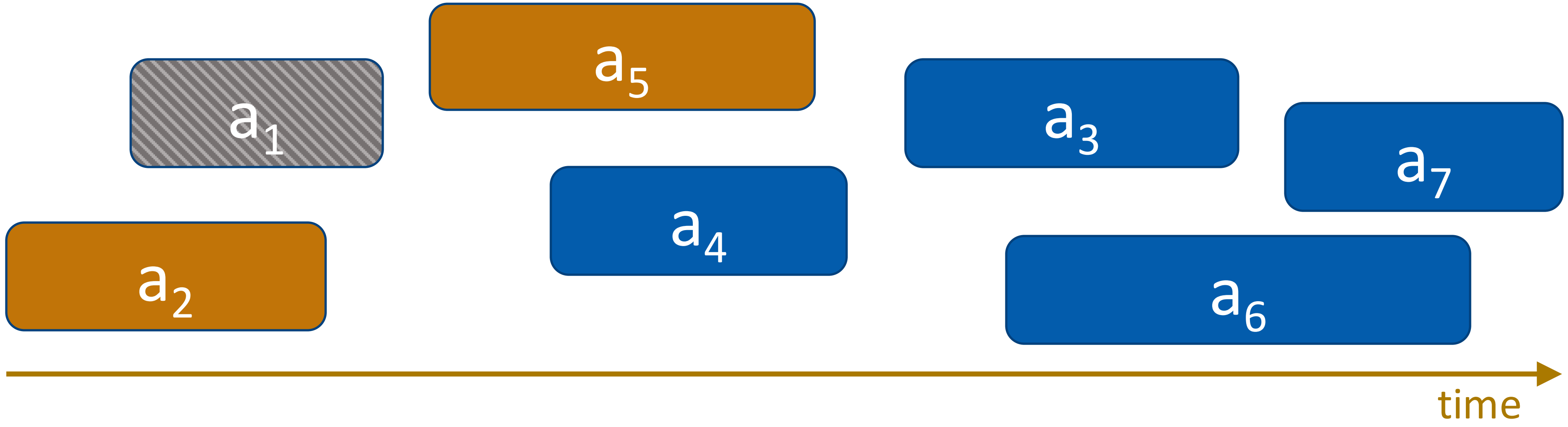

Greedy Algorithm

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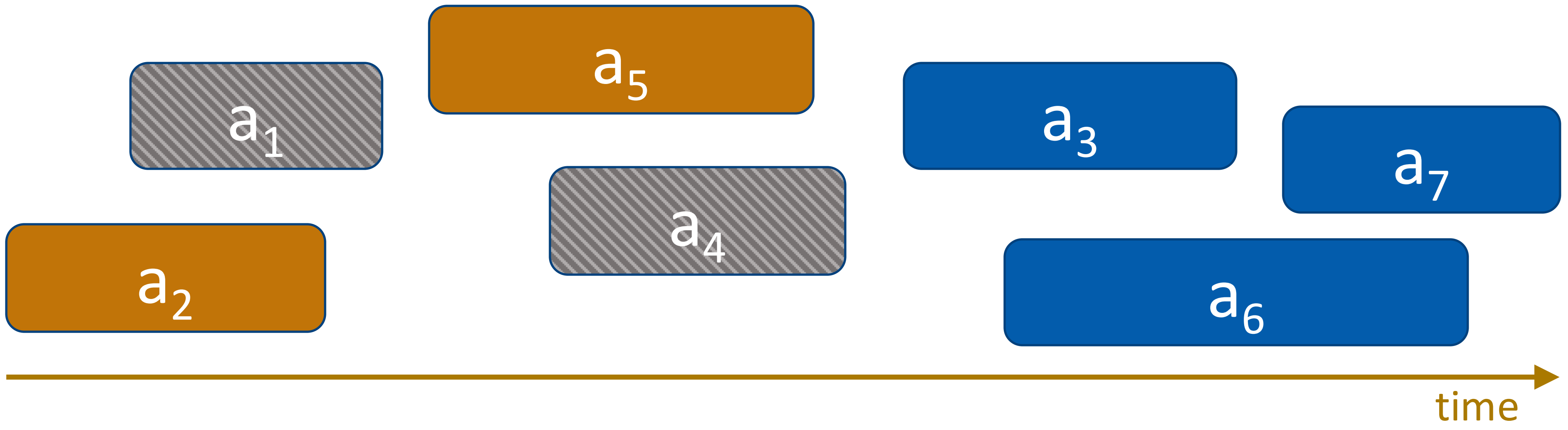

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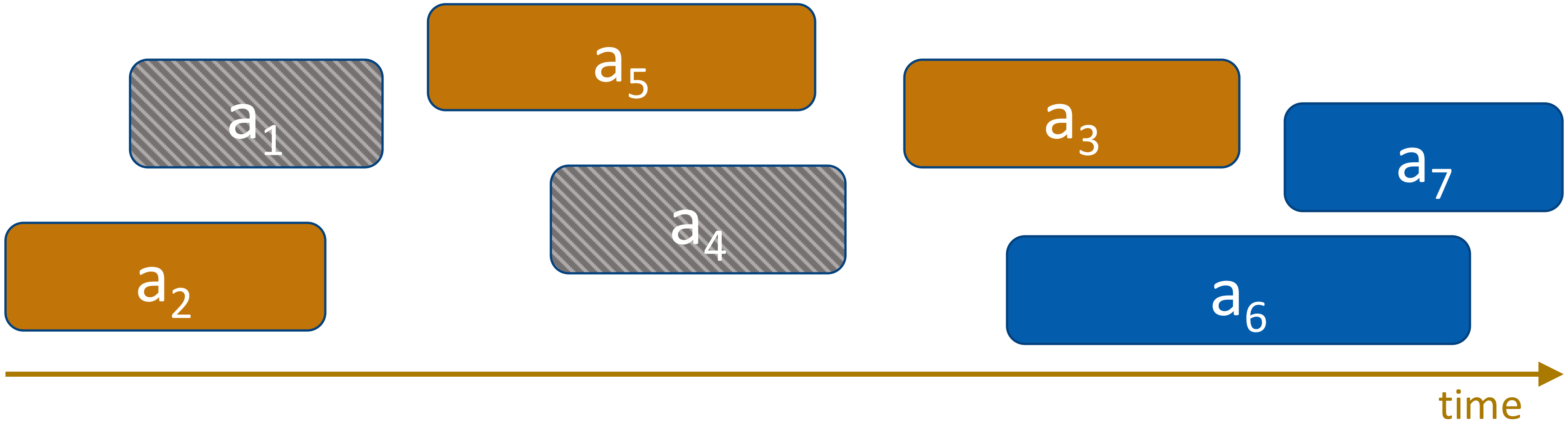

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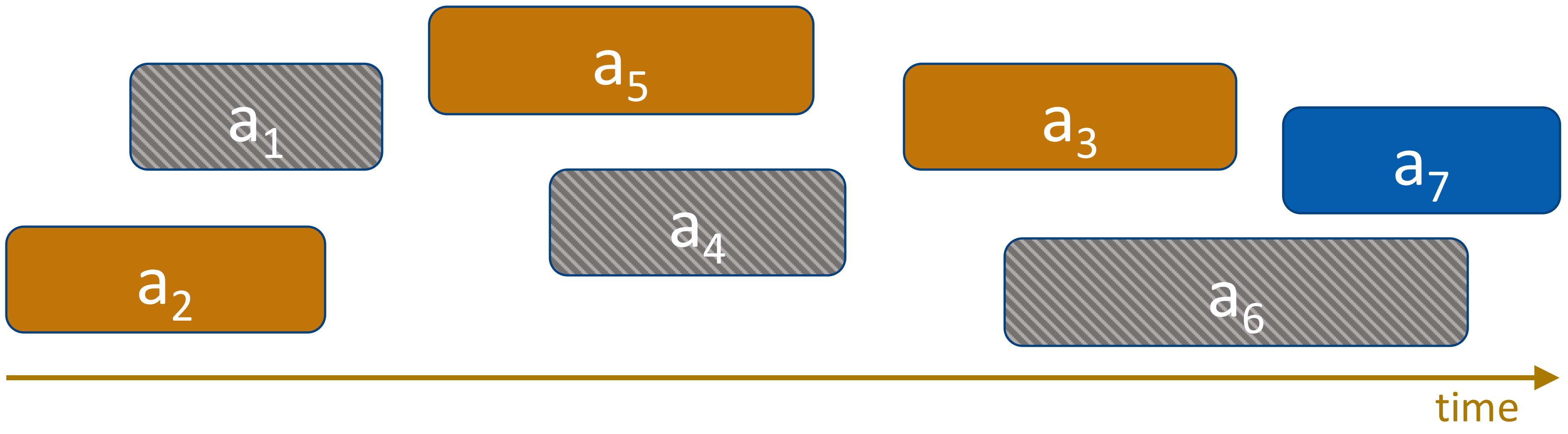

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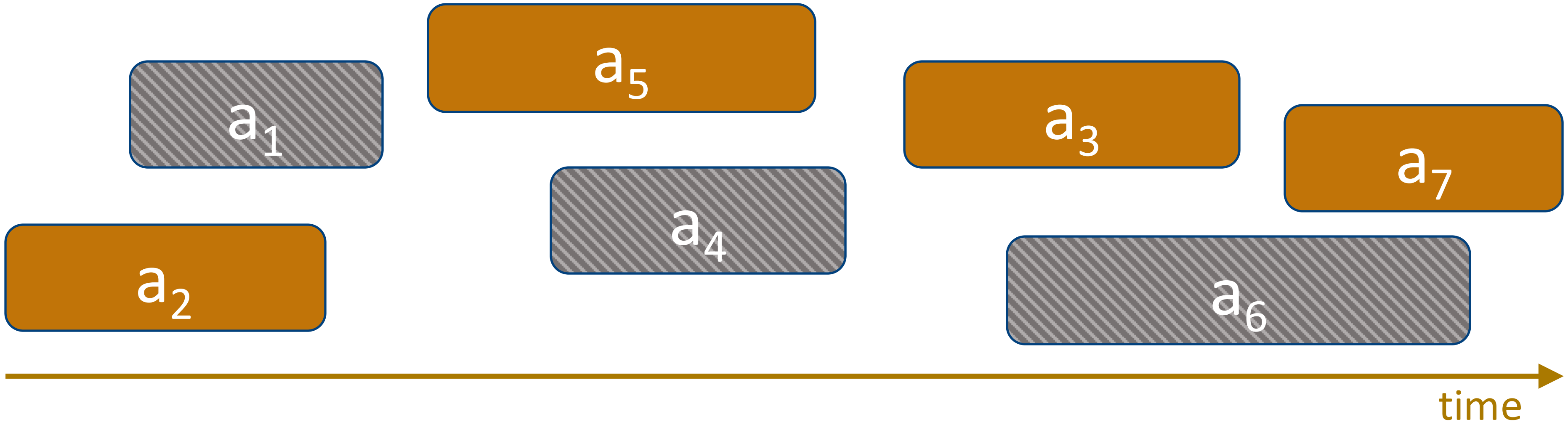


Efficiency

- Running time:
  - O(n) if the activities are already sorted by finish time.
  - Otherwise, O(n log(n)) if you have to sort them first.

Three Questions

1. Does this greedy algorithm for activity selection work?
   – Yes

2. Greedy is simple. But why are we getting to it in week 9 (not earlier)?
   – Proving that greedy algorithms work is often not so easy…

3. In general, when are greedy algorithms a good idea?
   – When the problem exhibits especially nice optimal substructure.

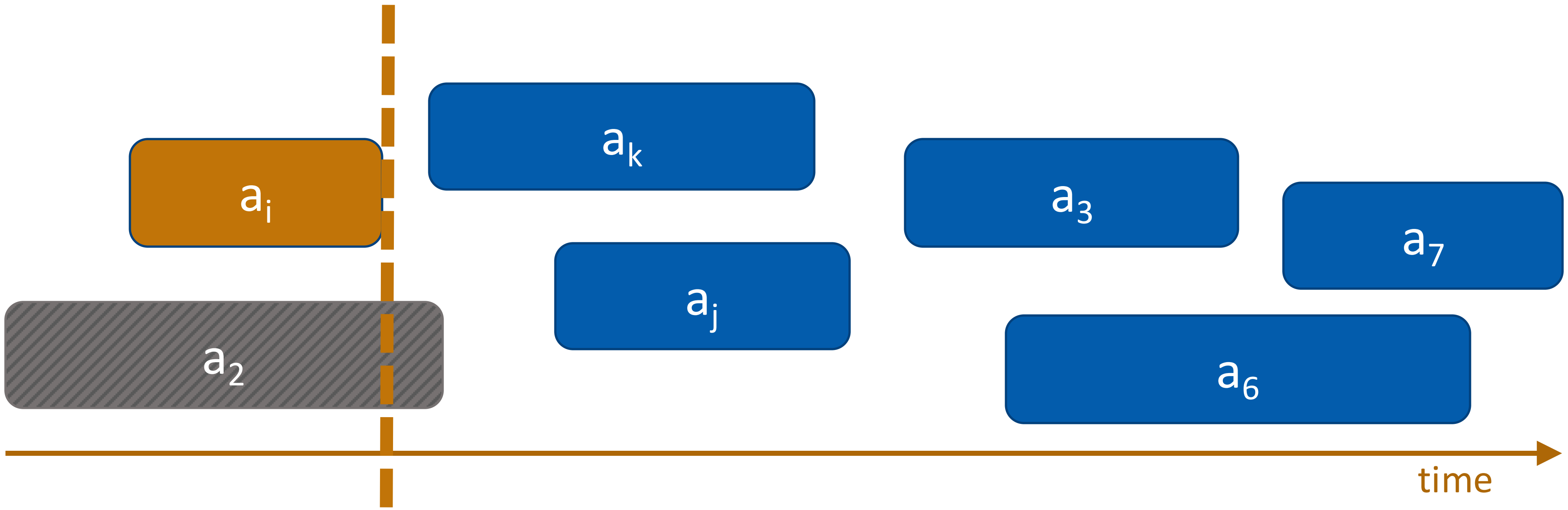
Back to Activity Selection

Why does it work?

- **We never rule out an optimal solution**
- At the end of the algorithm, we've got some solution.
- So it must be optimal.

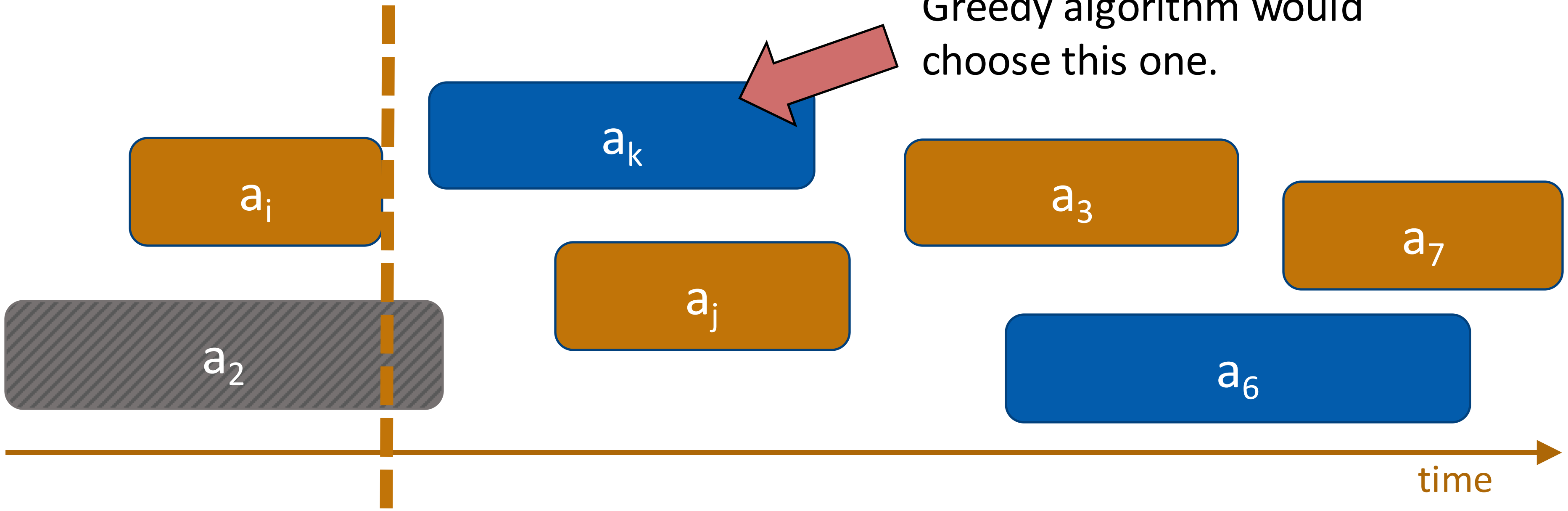
The Correctness of Activity Selection

- Suppose we've already chosen a_i, and there is still an optimal solution T* that extends our choices.



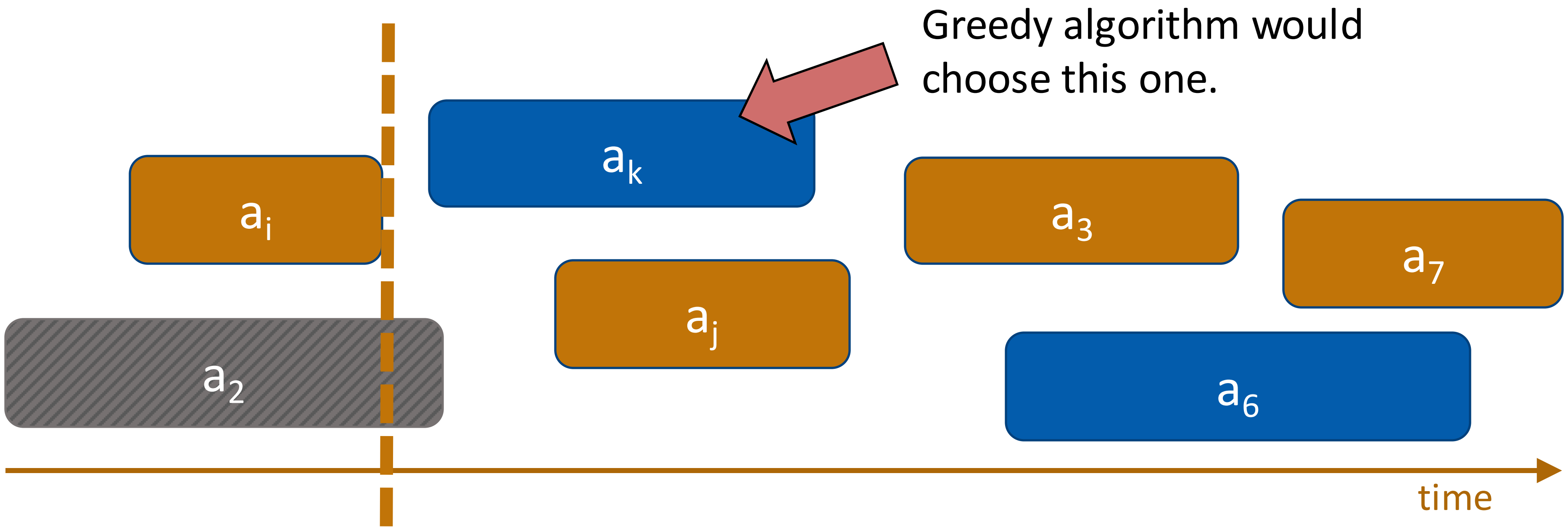
The Correctness of Activity Selection

- Suppose we've already chosen a_i, and there is still an optimal solution T* that extends our choices.
- Now consider the next choice we make, say it's a_k.
- If a_k is in T*, we're still on track.



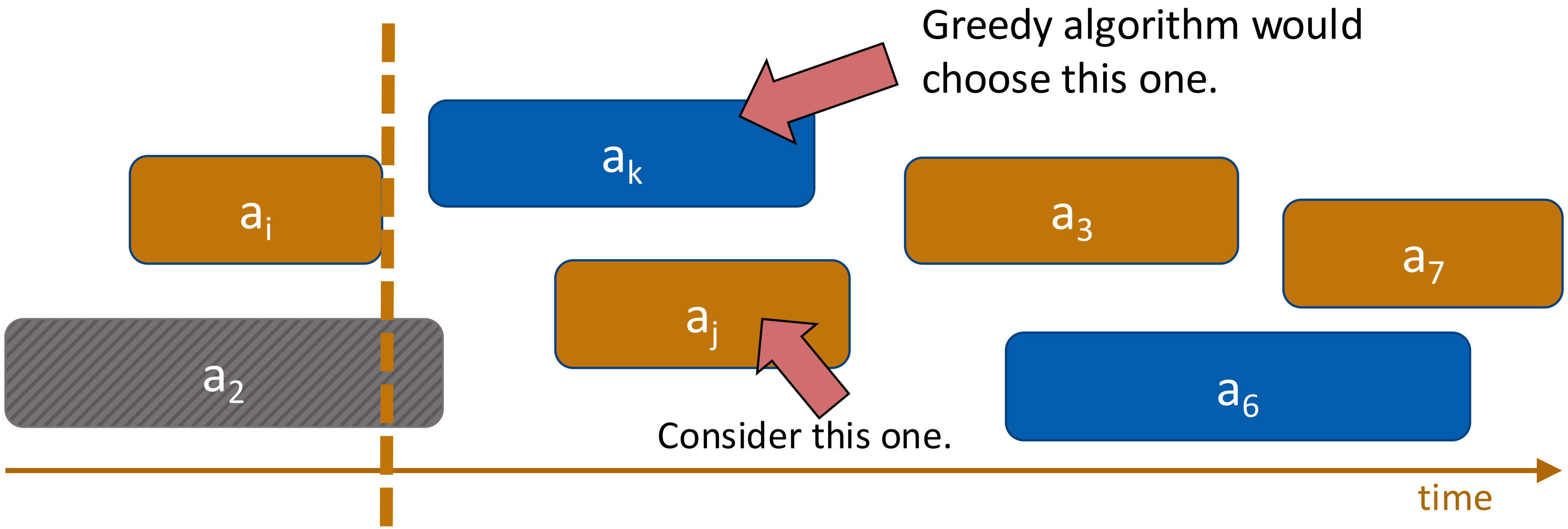
The Correctness of Activity Selection

- Suppose we've already chosen a_i, and there is still an optimal solution T* that extends our choices.
- Now consider the next choice we make, say it's a_k.
- If a_k is not in T*…



The Correctness of Activity Selection

- If a_k is **not** in T*…
- Let a_j be the activity in T* with the smallest end time.
- Now consider schedule T you get by swapping a_j for a_k



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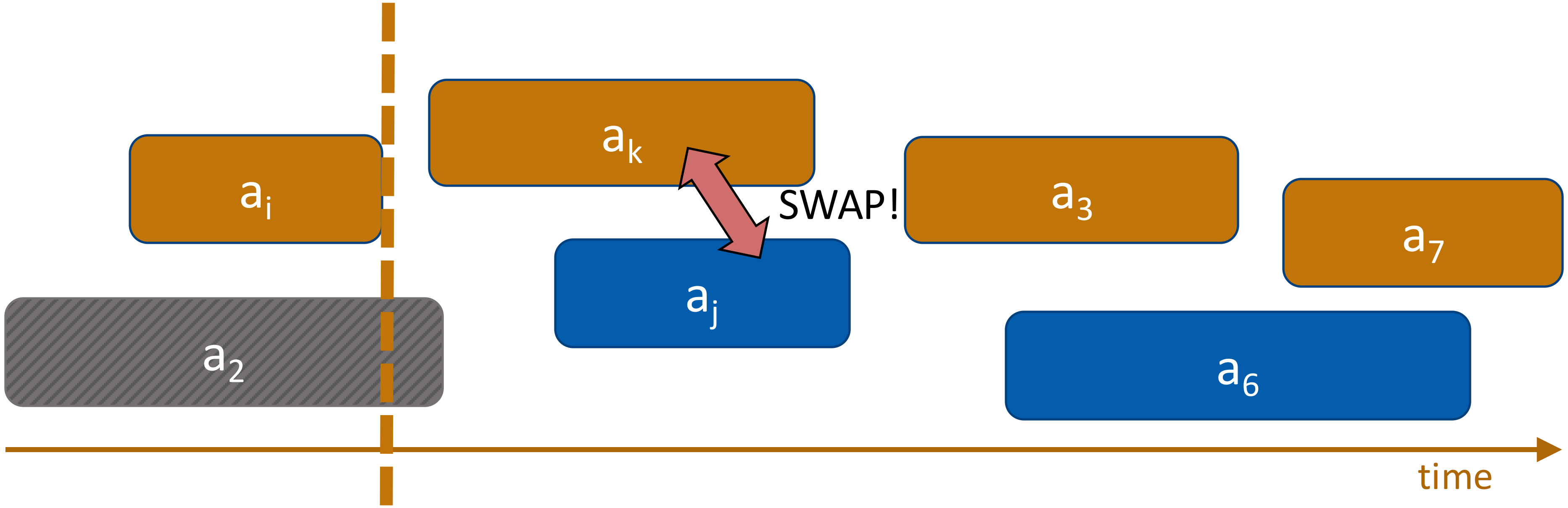
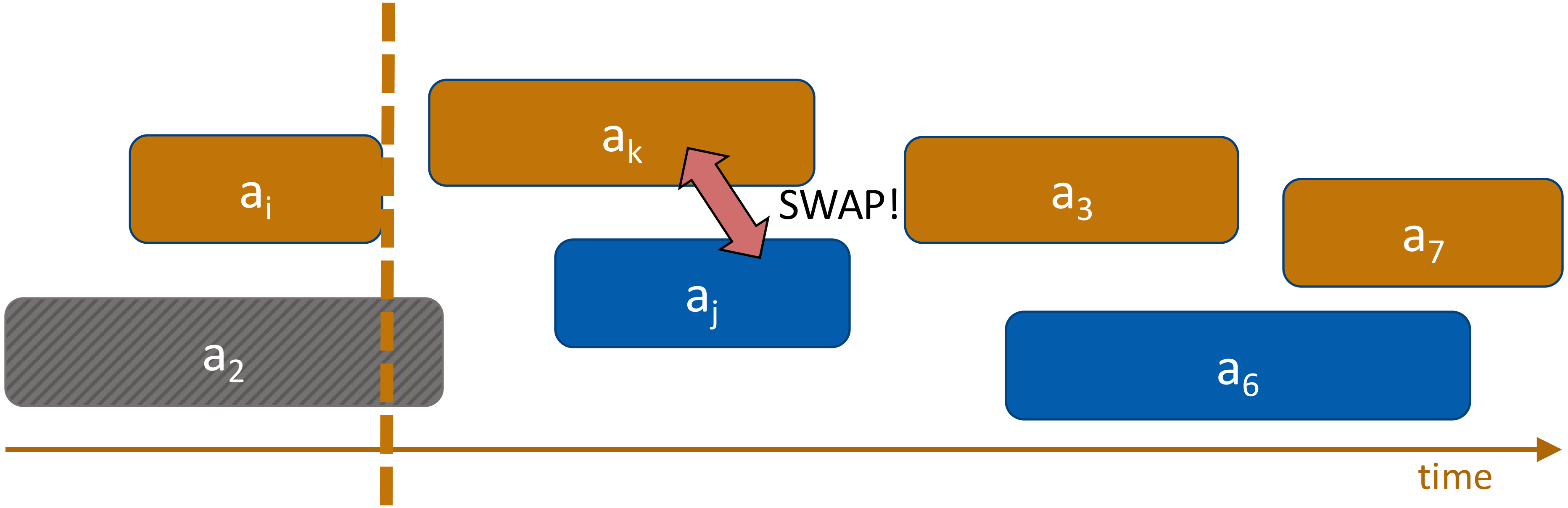


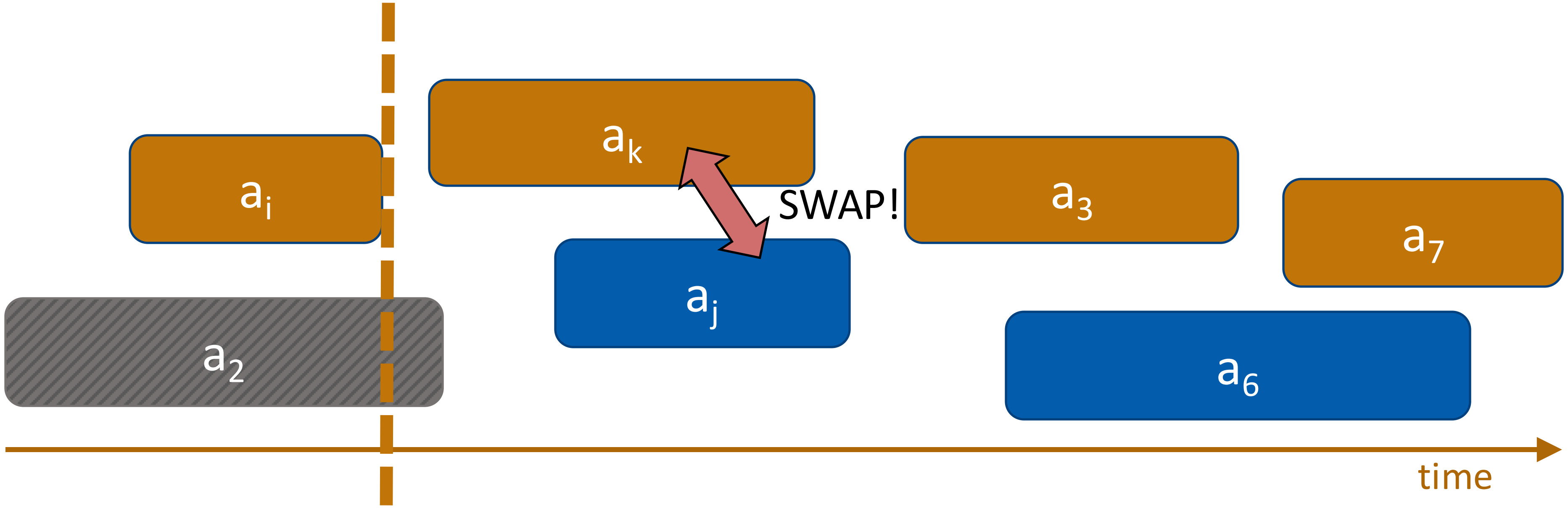
The Correctness of Activity Selection

- This schedule T is still allowed.
  - Since a_k has the smallest ending time, it ends before a_j.
  - Thus, a_k doesn't conflict with anything chosen after a_j.
- And T is still optimal.
  - It has the same number of activities as T*.



The Correctness of Activity Selection

- We've just shown:
  - If there was an optimal solution that extends the choices we made so far…
  - …then there is an optimal schedule that also contains our next greedy choice a_k



The Correctness of Activity Selection

So it's correct!

- **We never rule out an optimal solution**
- At the end of the algorithm, we've got some solution.
- So it must be optimal.

A Common Strategy

A common strategy for proving the correctness of greedy algorithms:

- Make a **series of choices**.
- Show that, at each step, our choice **won't rule out an optimal solution** at the end of the day.
- After we've made all our choices, we haven't ruled out an optimal solution, **so we must have found one.**

A Common Strategy

- Inductive Hypothesis:
  - After greedy choice t, you haven't ruled out success.
- Base case:
  - Success is possible before you make any choices.
- Inductive step:
  - If you haven't ruled out success after choice t, then you won't rule out success after choice t+1.
- Conclusion:
  - If you reach the end of the algorithm and haven't ruled out success then you must have succeeded.

A Common Strategy

A common strategy for showing we don't rule out the optimal solution:

- Suppose that you're on track to make an optimal solution T*.
  - E.g., after you've picked activity i, you're still on track.
- Suppose that T* *disagrees* with your next greedy choice.
  - E.g., it *doesn't* involve activity k.
- Manipulate T* in order to make a solution T that's not worse but that *agrees* with your greedy choice.
  - E.g., swap whatever activity T* did pick next with activity k.

Three Questions

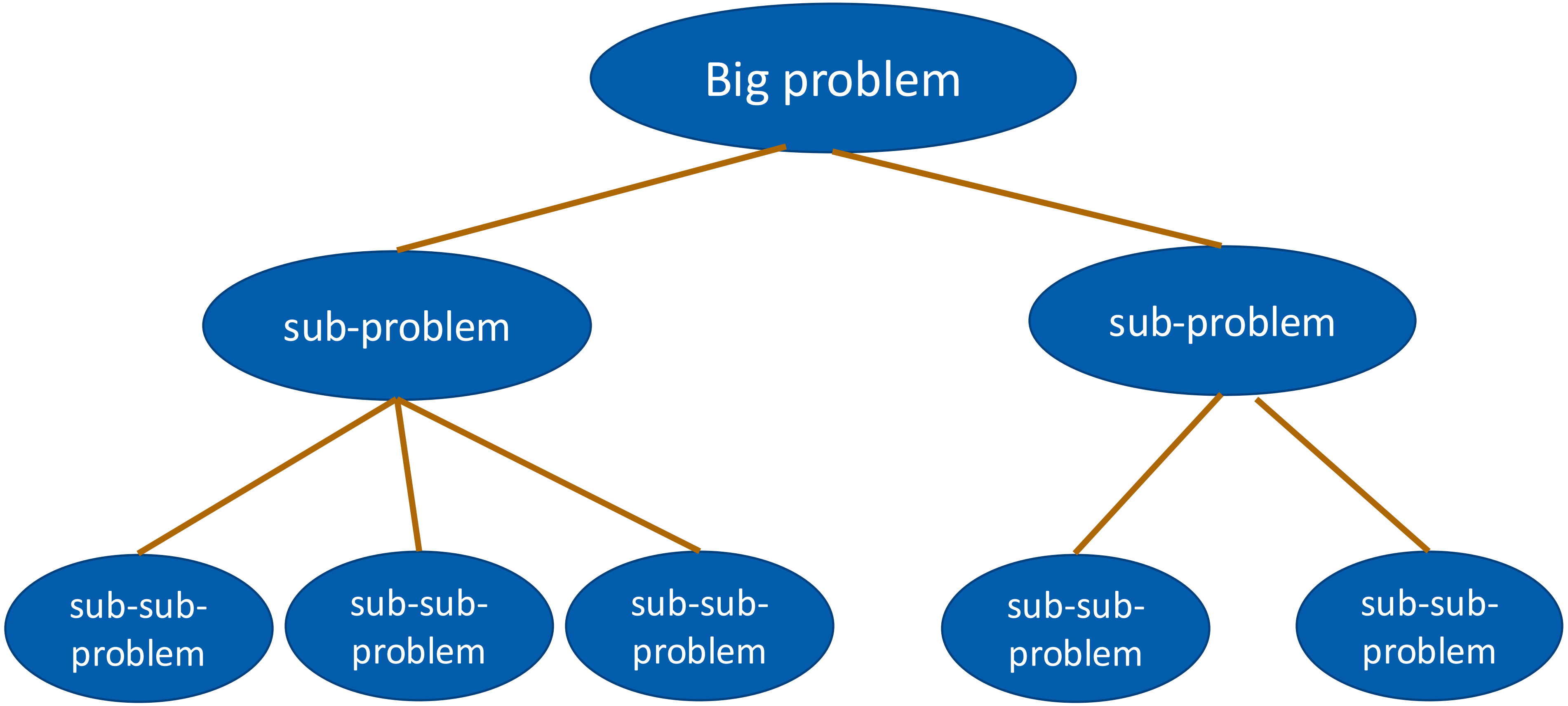
1. Does this greedy algorithm for activity selection work?
   – Yes

2. Greedy is simple. But why are we getting to it in week 9?
   – Proving that greedy algorithms work is often not so easy…

3. In general, when are greedy algorithms a good idea?
   – When the problem exhibits especially nice optimal substructure.

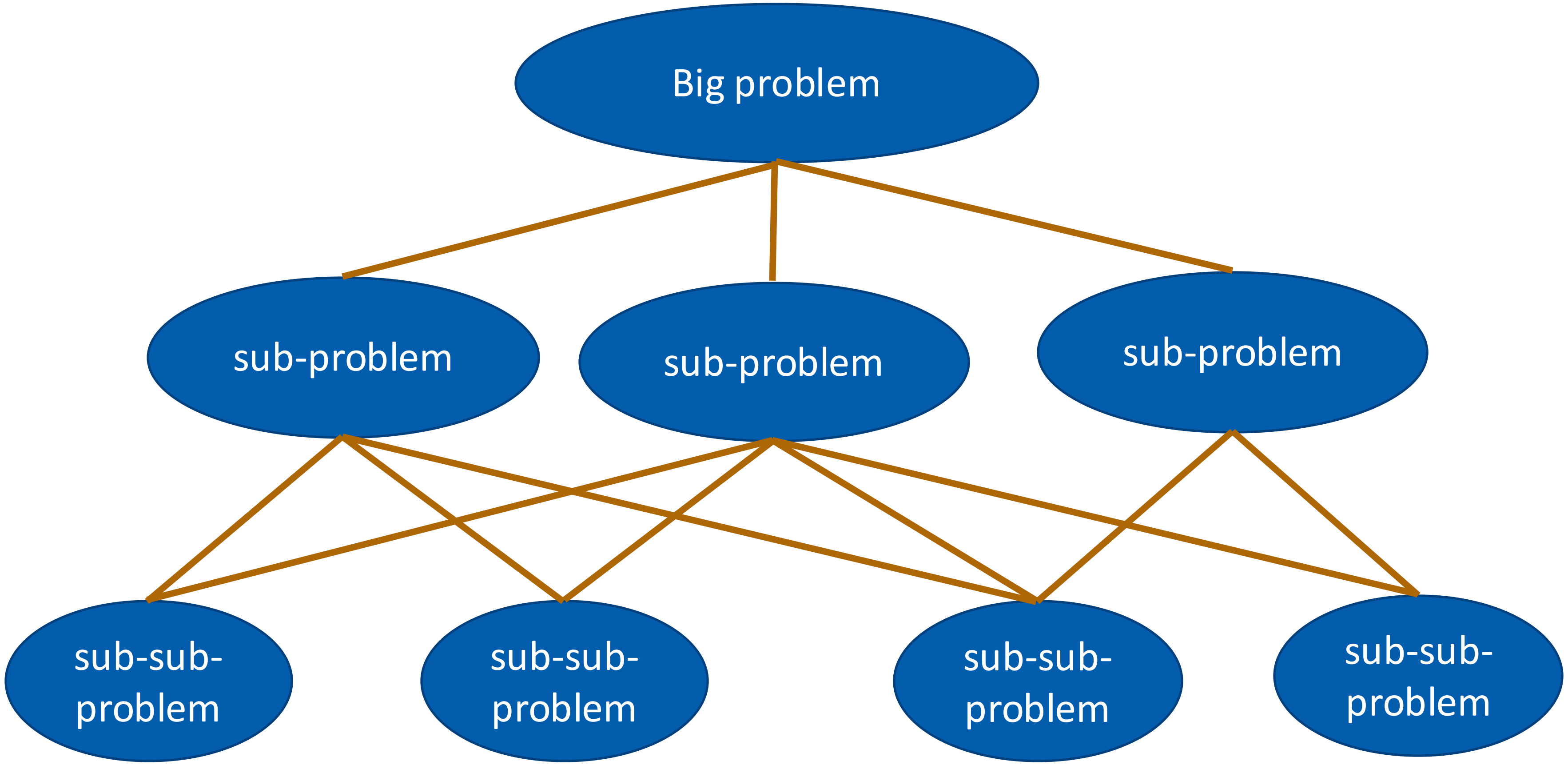
Sub-problem graph view

- Divide-and-conquer:



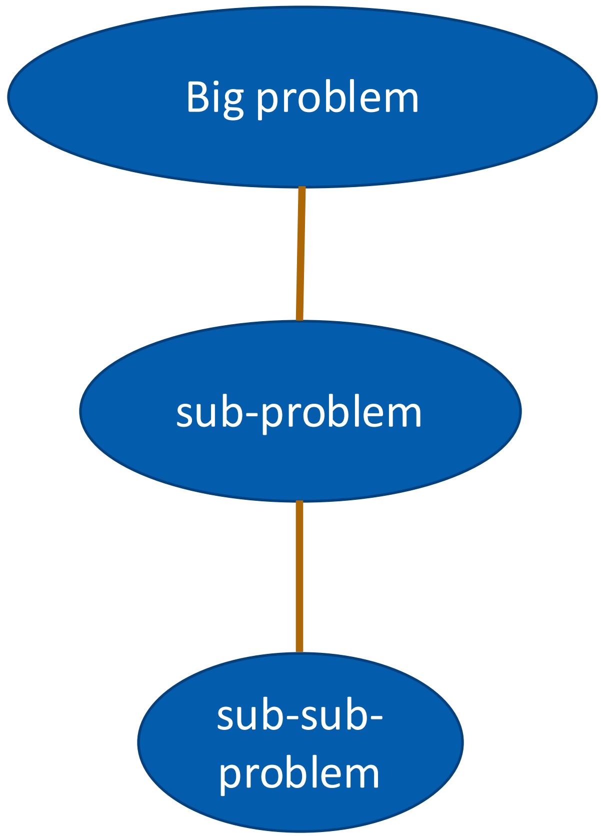
Sub-problem graph view

- Dynamic Programming:



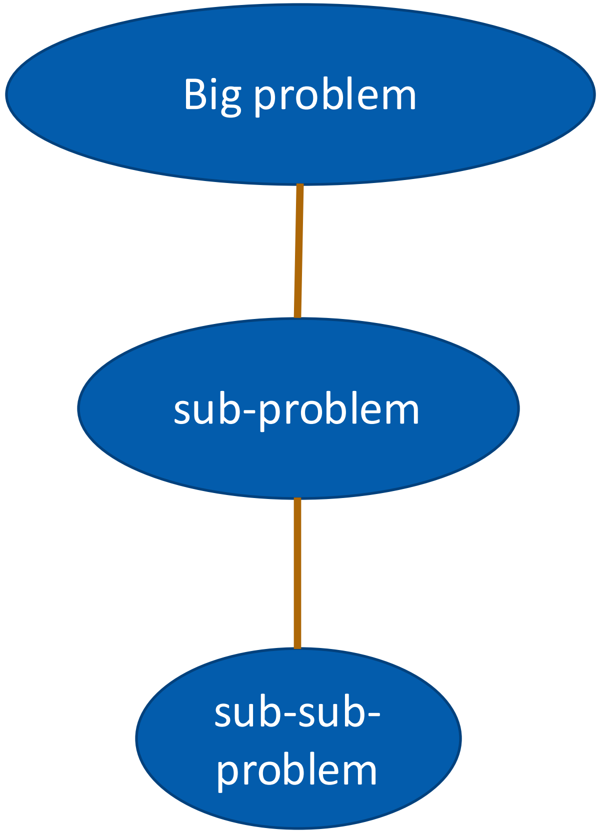
Sub-problem graph view

- Greedy algorithms:



Sub-problem graph view

- Greedy algorithms:



- Not only is there **optimal sub-structure:**
  - optimal solutions to a problem are made up from optimal solutions of sub-problems
- but each problem **depends on only one sub-problem**.

Three Questions

1. Does this greedy algorithm for activity selection work?
   – Yes ✓

2. Greedy is simple. But why are we getting to it in week 9? ✓
   – Proving that greedy algorithms work is often not so easy…

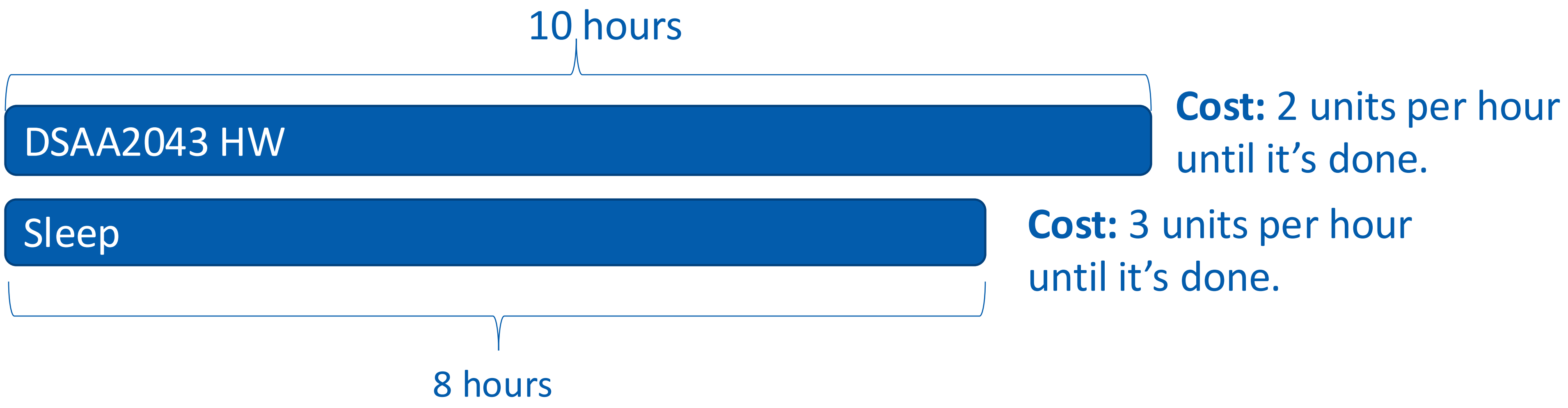
3. In general, when are greedy algorithms a good idea? ✓
   – When the problem exhibits especially nice optimal substructure.

Another Example: Scheduling

- DSAA2043 HW
- Personal hygiene
- Math HW
- Administrative stuff for student club
- Econ HW
- Do laundry
- Sports
- Practice musical instrument
- Read lecture notes
- Have a social life
- Sleep

Scheduling

- n tasks
- Task i takes t_i hours
- For every hour that passes until task i is done, pay c_i



- DSAA2043 HW, then Sleep: costs **$10 \cdot 2 + (10 + 8) \cdot 3 = 74$ units**
- Sleep, then DSAA2043 HW: costs **$8 \cdot 3 + (10 + 8) \cdot 2 = 60$ units**

Scheduling

- This problem breaks up nicely into sub-problems:

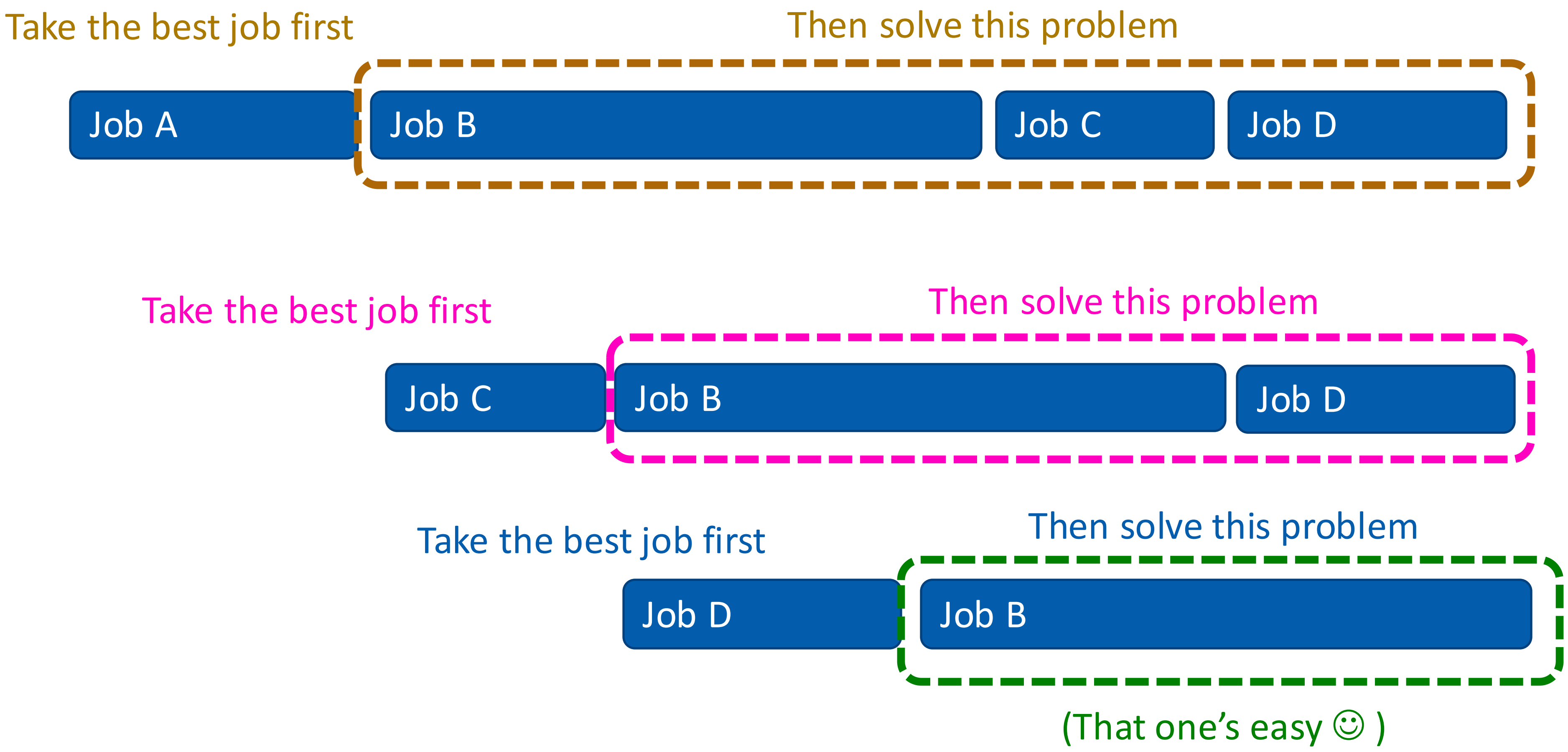
Suppose this is the optimal schedule:

Then this must be the optimal schedule on just jobs B,C,D.

If not, then rearranging B,C,D could make a better schedule than (A,B,C,D)!

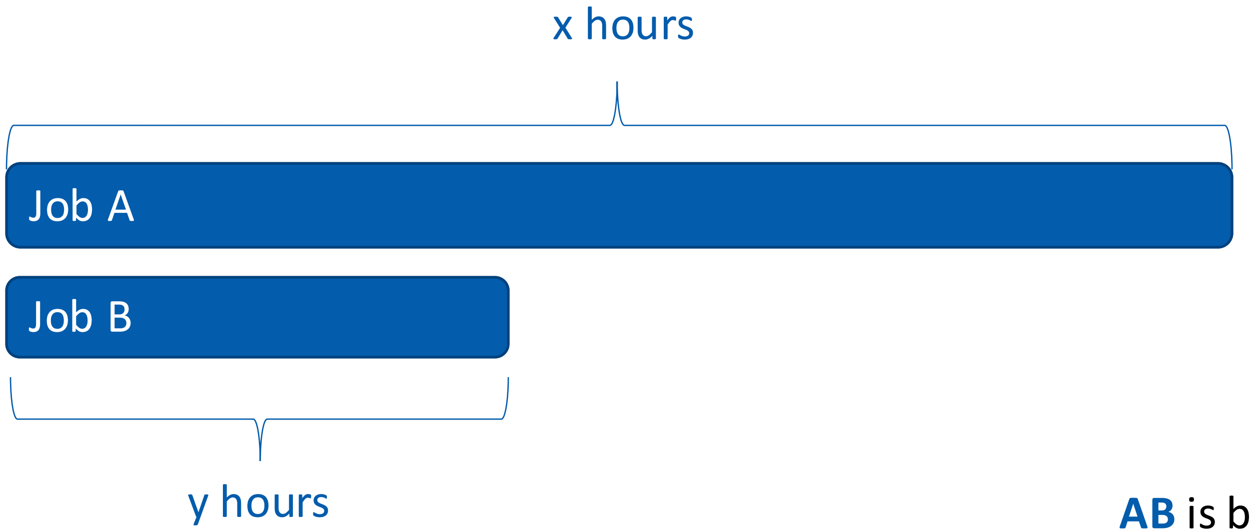
Scheduling

- Seems amenable to a greedy algorithm:



What does “best” mean?

- Of these two jobs, which should we do first?



Cost: z units per hour until it’s done.

Cost: w units per hour until it’s done.

- Cost(**A then B**) = x · z + (x + y) · w
- Cost(**B then A**) = y · w + (x + y) · z

AB is better than **BA** when:

$$xz + (x+y)w \leq yw + (x+y)z$$
$$xz + xw + yw \leq yw + xz + yz$$
$$wx \leq yz$$
$$\frac{w}{y} \leq \frac{z}{x}$$

Idea for Greedy

- Choose the job with the biggest $\frac{\text{cost of delay}}{\text{time it takes}}$ ratio.

Correctness

- Suppose you have already chosen some jobs, and haven’t yet ruled out success:

Job E | Job C | Job A | Job B | Job D

- Then if you choose the next job to be the one left that maximizes the ratio **cost/time**, you still won’t rule out success.
- **Proof sketch:** Exchange Argument
  - Say Job B maximizes this ratio, but it’s not the next job in the opt. soln.
  - Switch A and B! Nothing else will change, and we just showed that the cost of the solution won’t increase.

Job E | Job C | Job B | Job A | Job D

  - Repeat until B is first.

Job E | Job B | Job C | Job A | Job D

  - Now this is an optimal schedule where B is first.

Correctness

- Inductive Hypothesis:
  - After greedy choice t, you haven't ruled out success.
- Base case:
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- Inductive step:
  - If you haven't ruled out success after choice t, then you won't rule out success after choice t+1.
- Conclusion:
  - If you reach the end of the algorithm and haven't ruled out success then you must have succeeded.

Greedy Scheduling Solution

- **scheduleJobs**(JOBS):
  - Sort JOBS in decreasing order by the ratio:
    - $r_i = \frac{c_i}{t_i} = \frac{\text{cost of delaying job i}}{\text{time job i takes to complete}}$
  - **Return** JOBS

Running time: O(n log(n))

Minimum Spanning Trees

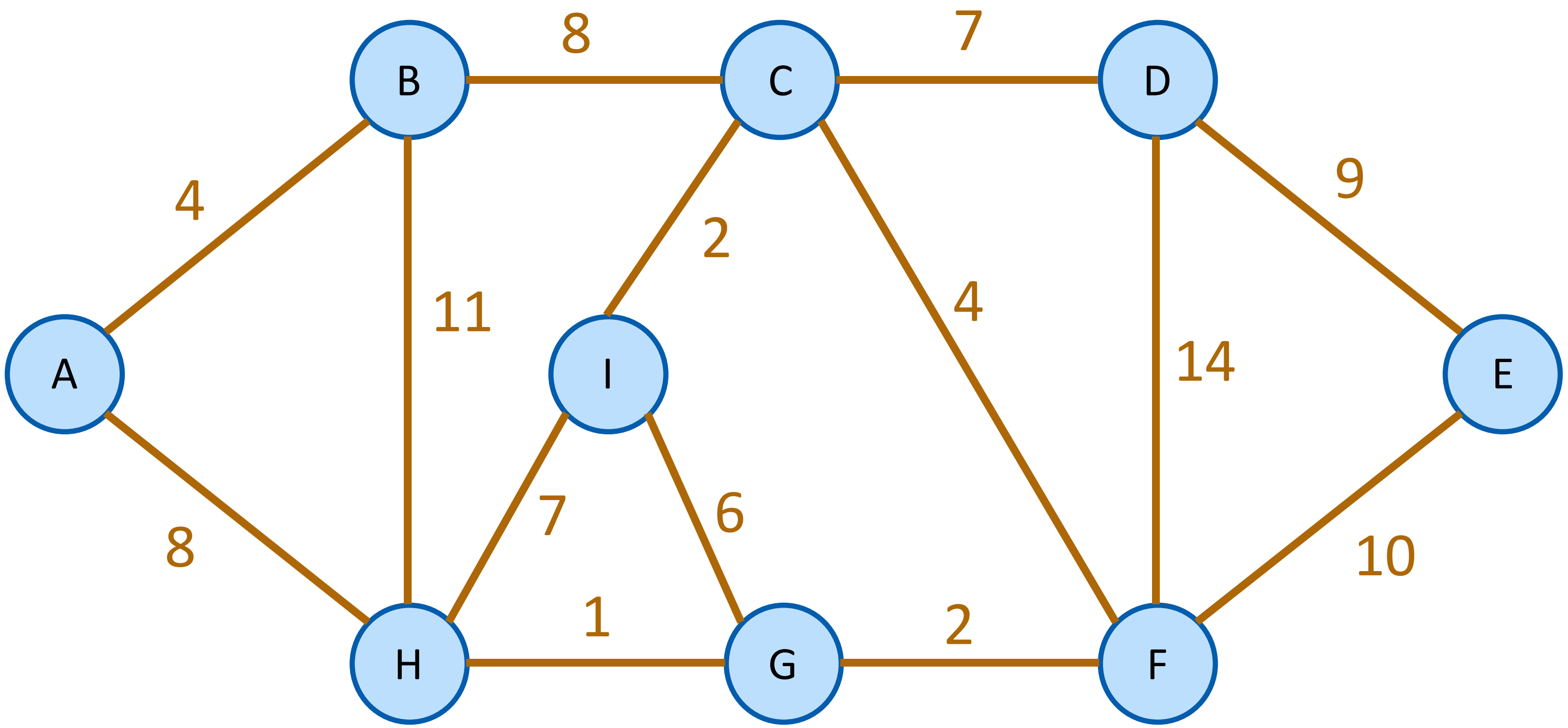
Minimum Spanning Trees

- Greedy algorithms for Minimum Spanning Tree.

- Agenda:
  1. What is a Minimum Spanning Tree?
  2. Short break to introduce some graph theory tools
  3. Prim's algorithm
  4. Kruskal's algorithm

Minimum Spanning Trees

- Say we have an undirected weighted graph



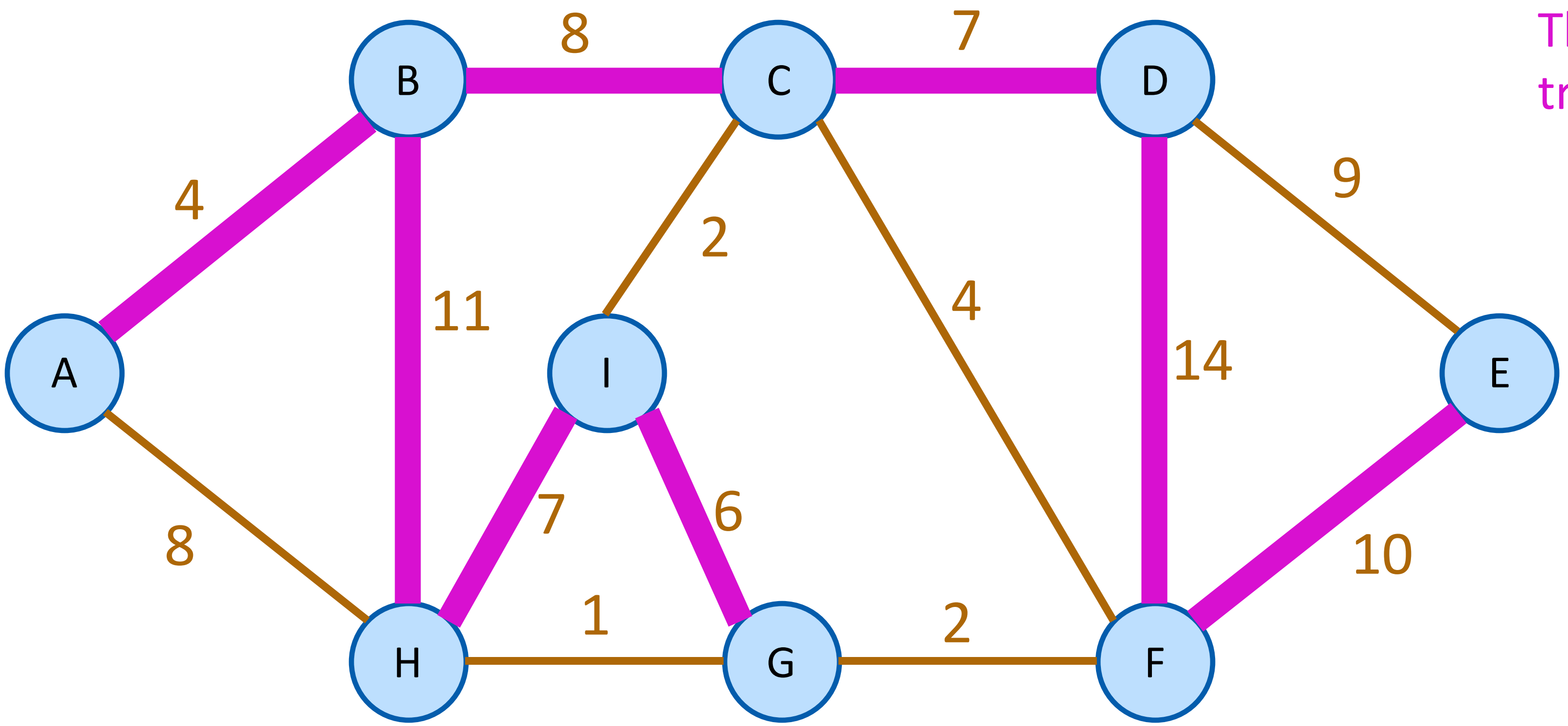
A **spanning tree** is a **tree** that connects all of the vertices.

A **tree** is a connected graph with no cycles!

Minimum Spanning Trees

- Say we have an undirected weighted graph

The **cost** of a spanning tree is the sum of the weights on the edges.

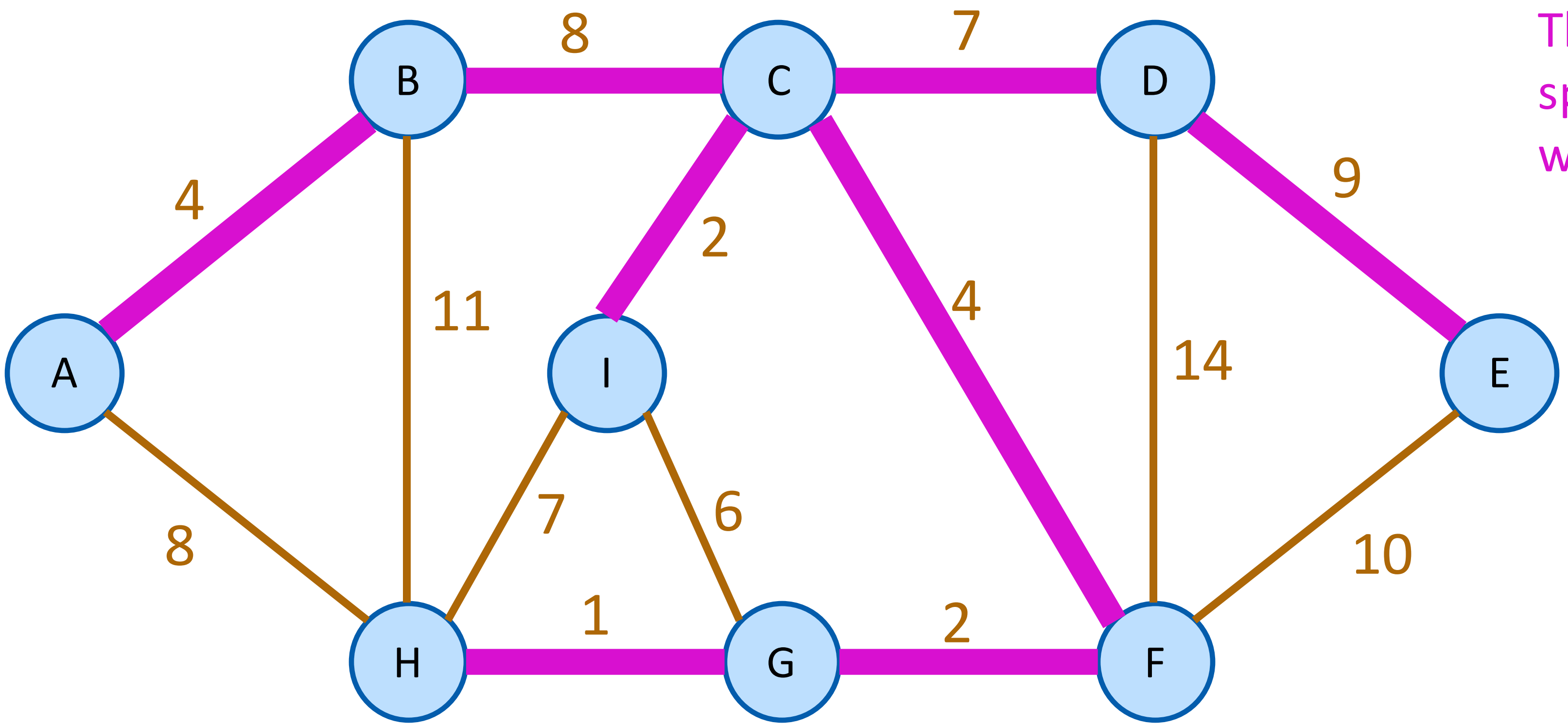


This is a spanning tree with cost 67.

A **spanning tree** is a **tree** that connects all of the vertices.

Minimum Spanning Trees

- Say we have an undirected weighted graph

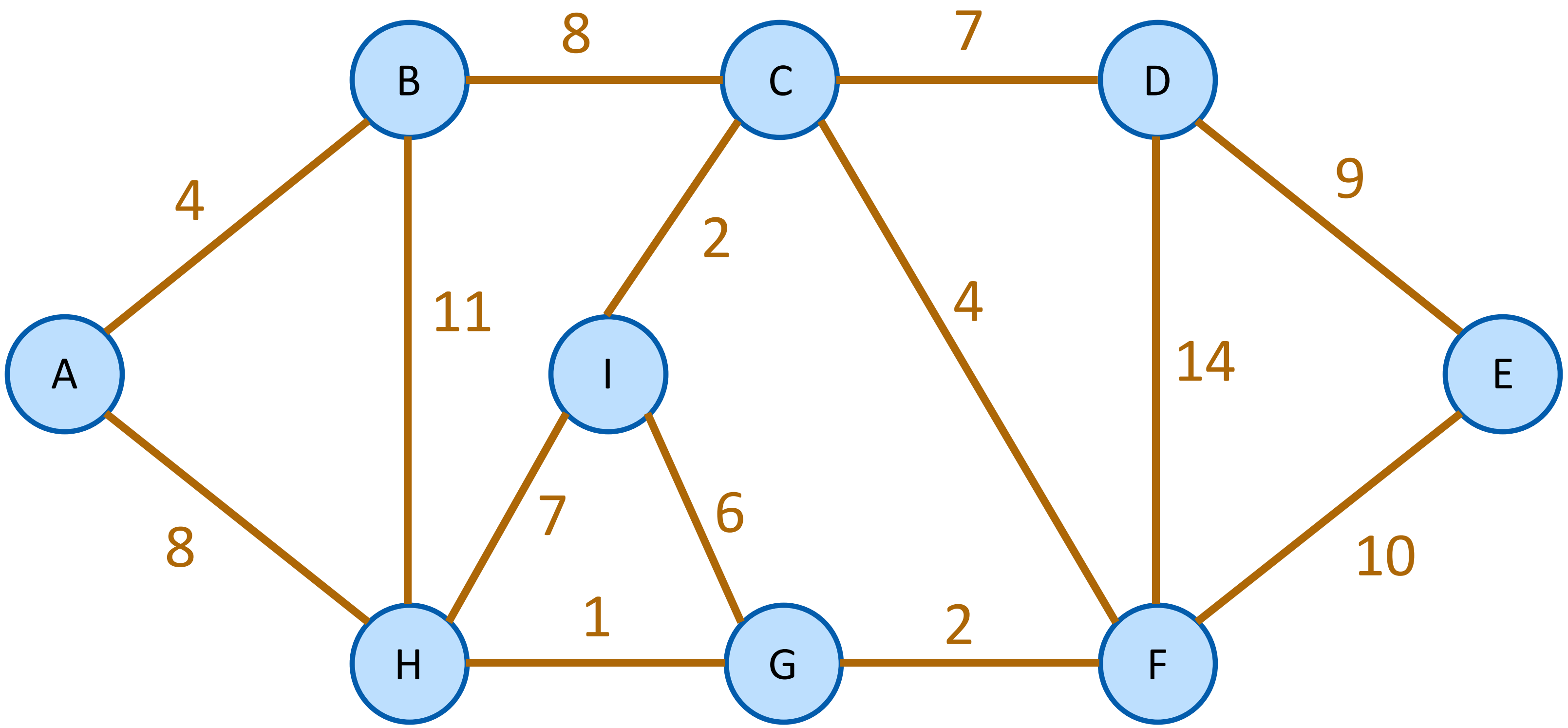


This is also a spanning tree, with cost 37.

A **spanning tree** is a **tree** that connects all of the vertices.

Minimum Spanning Trees

- Say we have an undirected weighted graph



minimum

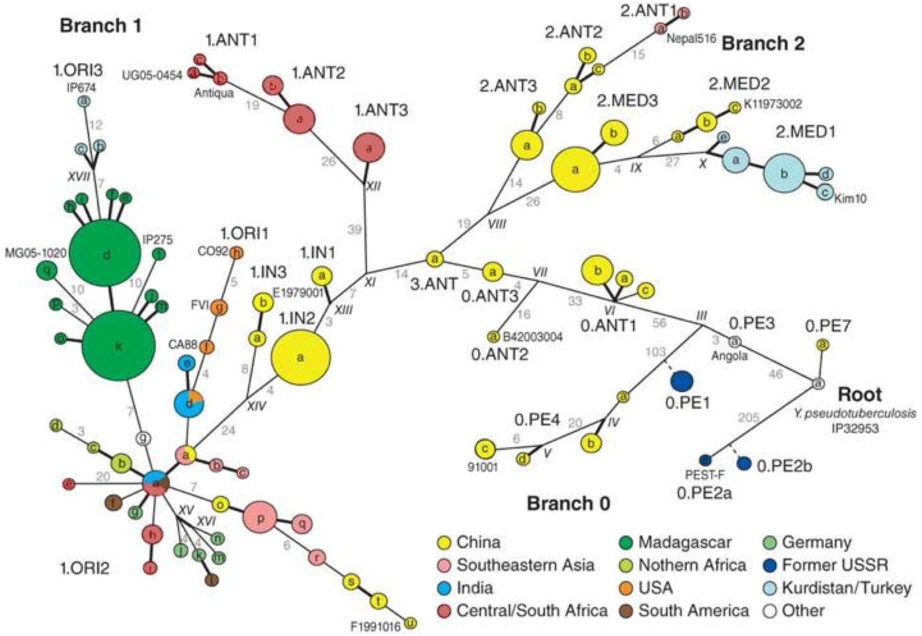
of minimum cost

A **spanning tree** is a **tree** that connects all of the vertices.

General def: tree that connects ONLY to a GIVEN subset of vertices

Why MSTs?

- Network design
  - Connecting cities with roads/electricity/telephone/…
- Cluster analysis
  - E.g., genetic distance
- Image processing
  - E.g., image segmentation
- Useful primitive
  - For other graph algs



How to find an MST

- Today we'll see two greedy algorithms.
- In order to prove that these greedy algorithms work, we'll show something like:

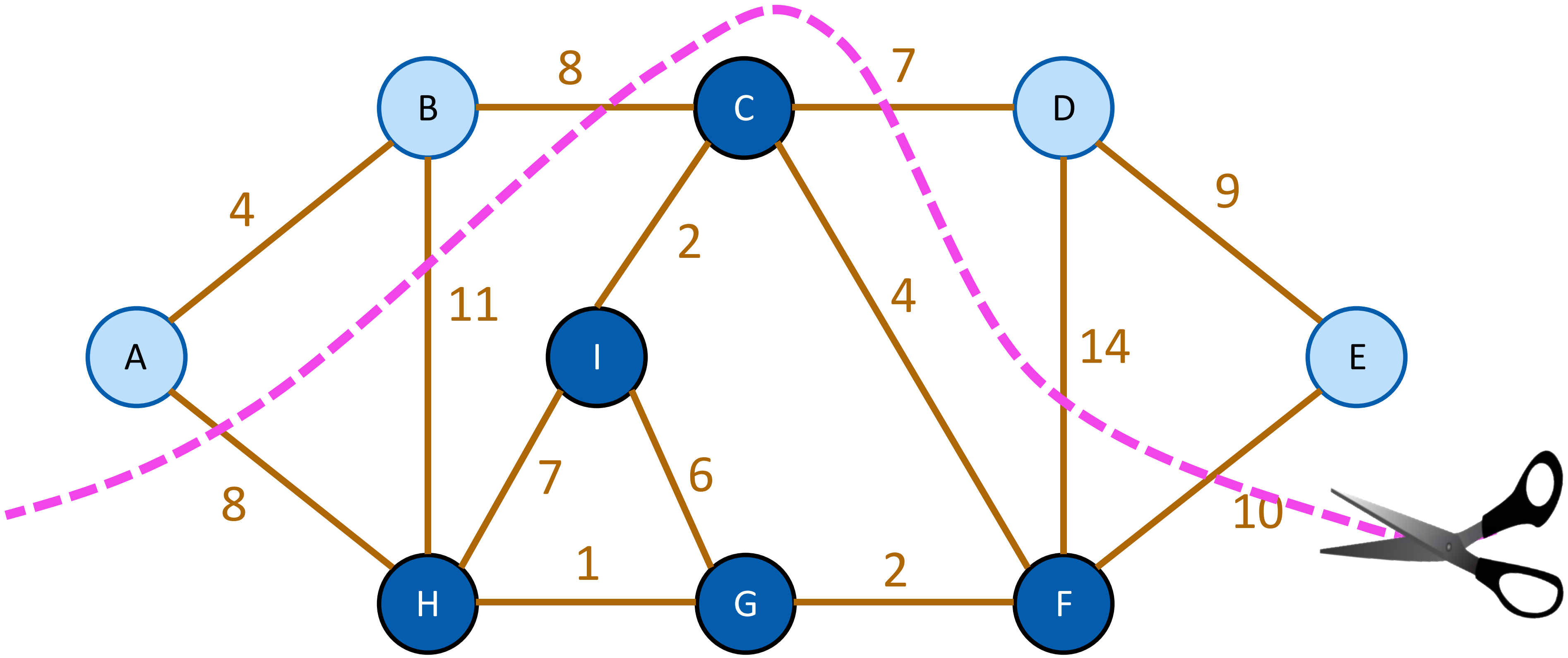
Suppose that our choices so far
are consistent with an MST.

Then the next greedy choice that we make
is still consistent with an MST.

- This is not the only way to prove that these algorithms work!

Brief Aside – Cuts in Graphs

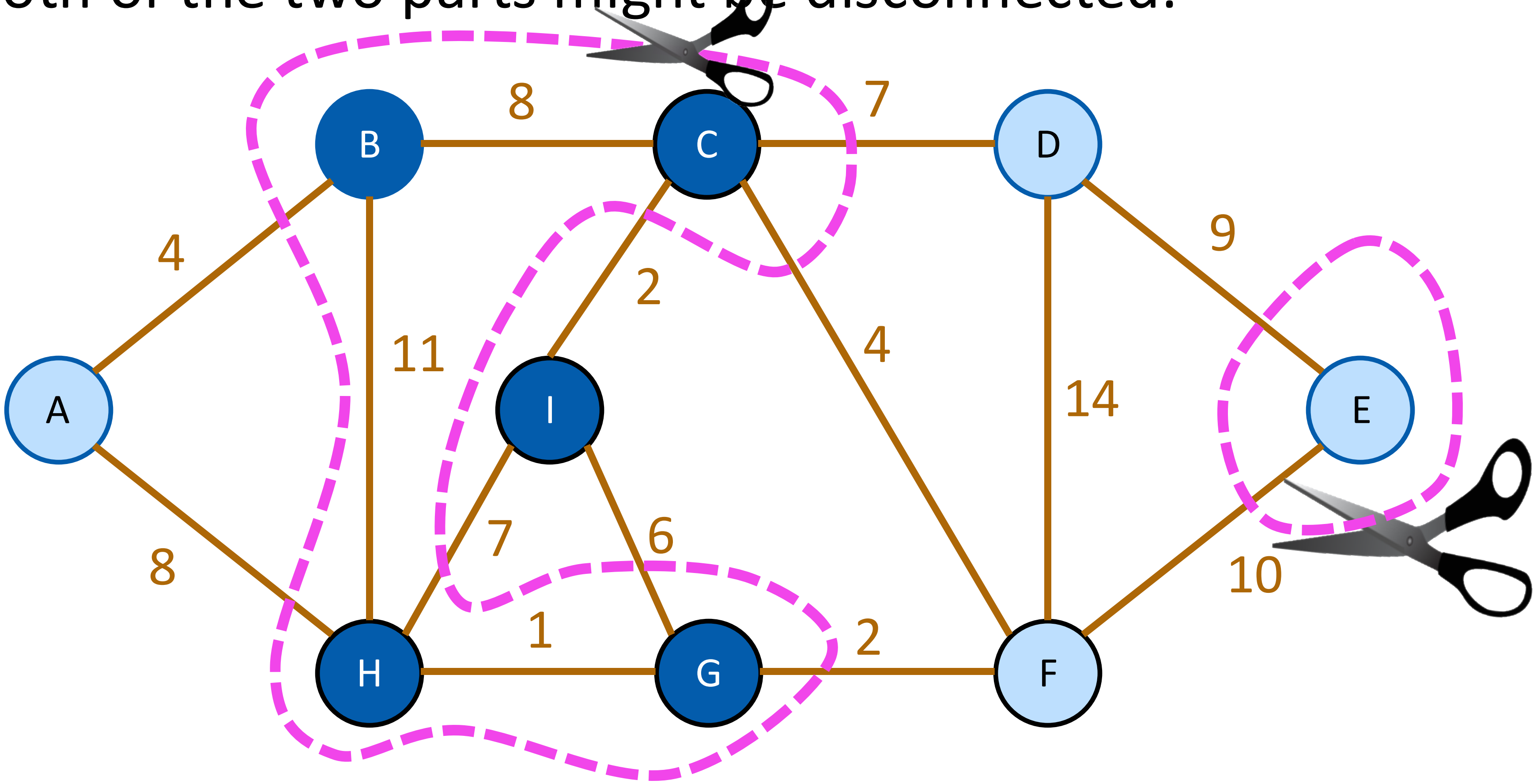
- A cut is a partition of the vertices into two parts:



This is the cut **"{A,B,D,E} and {C,I,H,G,F}"**

Brief Aside – Cuts in Graphs

- One or both of the two parts might be disconnected.

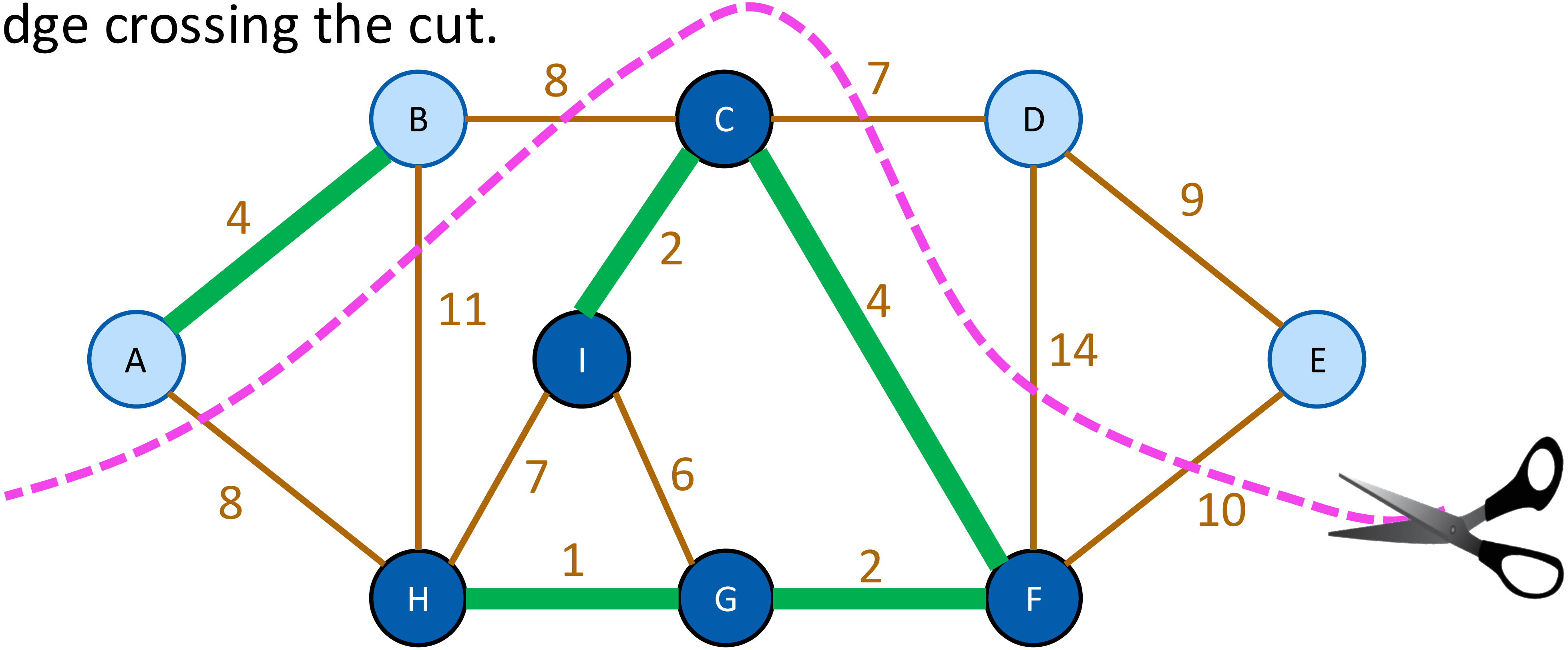


This is the cut **“{B,C,E,G,H} and {A,D,I,F}”**

Brief Aside – Cuts in Graphs

Let S be a set of edges in G

- We say a cut **respects** S if no edges in S cross the cut.
- An edge crossing a cut is called **light** if it has the smallest weight of any edge crossing the cut.



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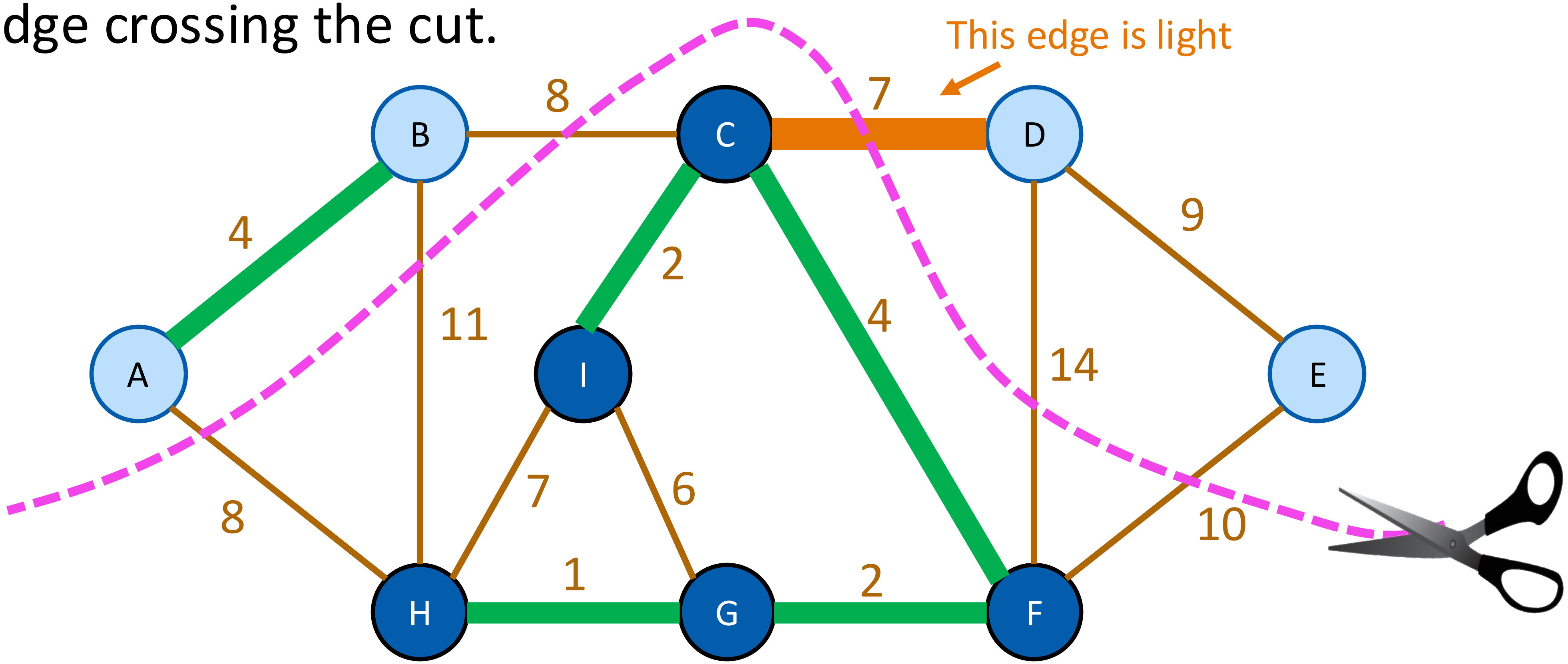


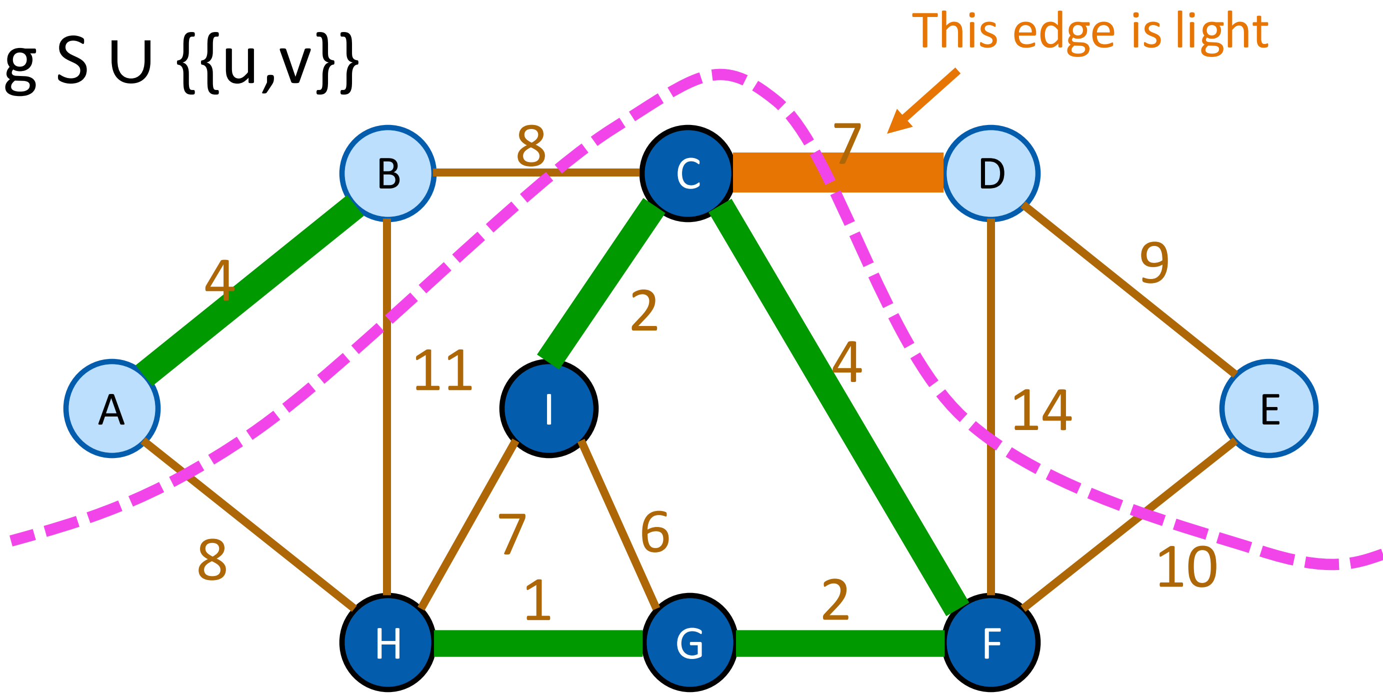
Brief Aside – Cuts in Graphs

Lemma

- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
- Let {u,v} be a light edge.
- Then there is an MST containing S U {{u,v}}

Aka:

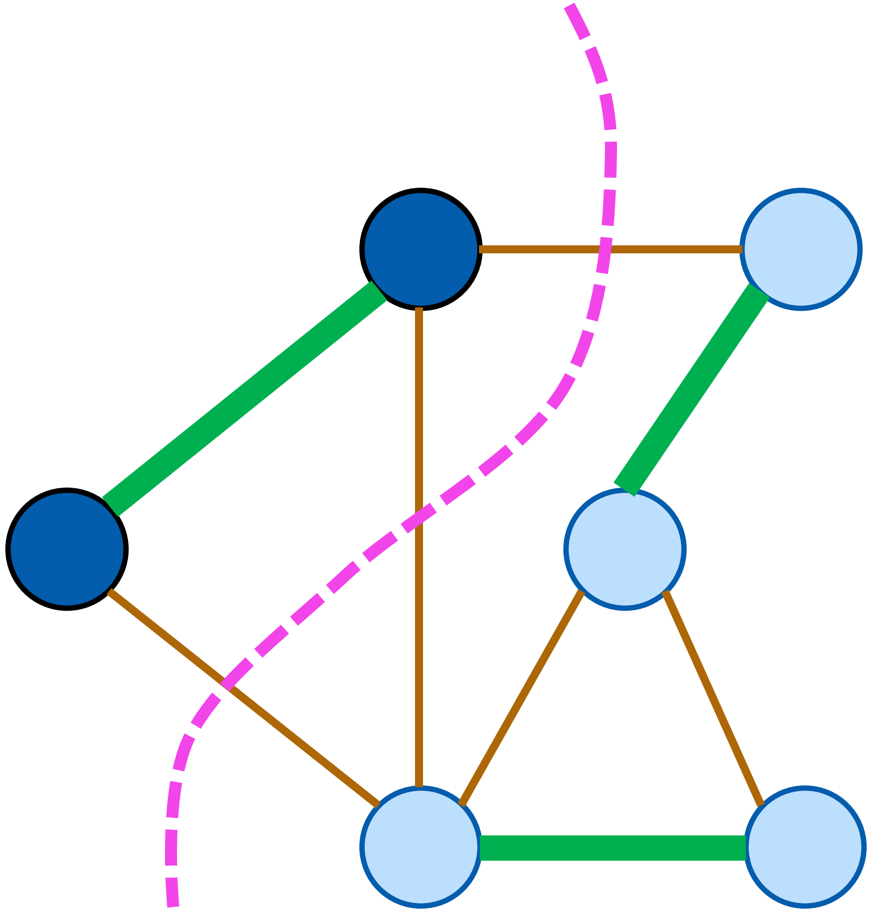
If we haven't ruled out the possibility of success so far, then adding a light edge still won't rule it out.



Brief Aside – Cuts in Graphs

Proof of Lemma

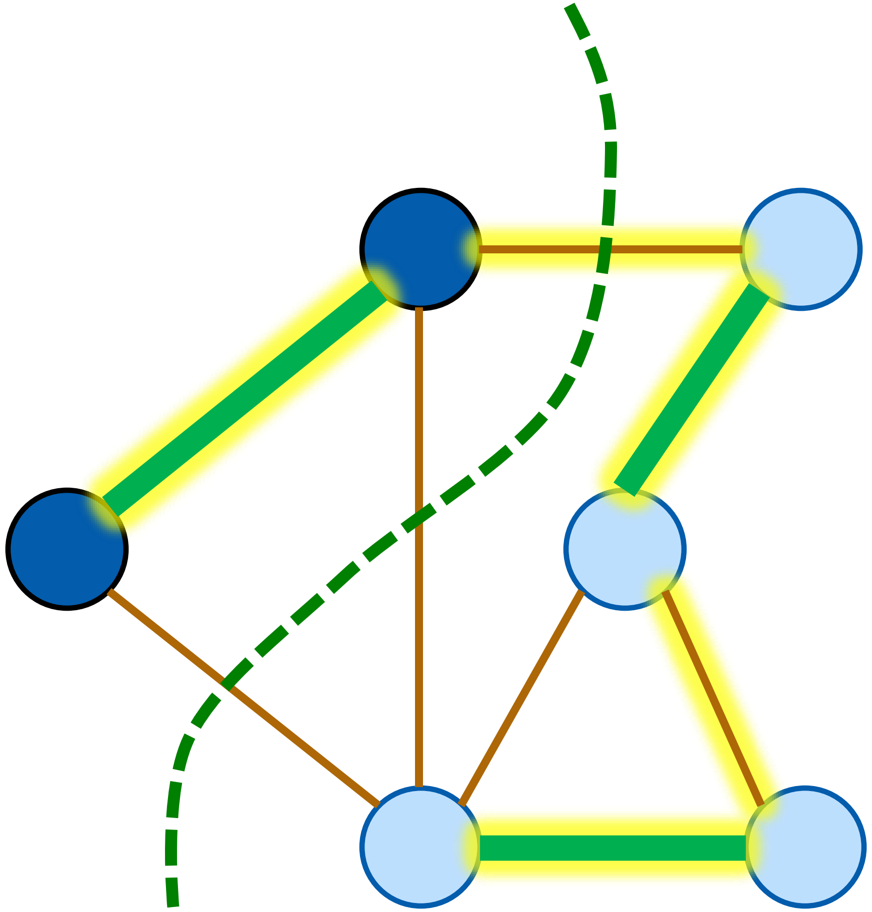
- Assume that we have:
  - a **cut** that respects **S**



Brief Aside – Cuts in Graphs

Proof of Lemma

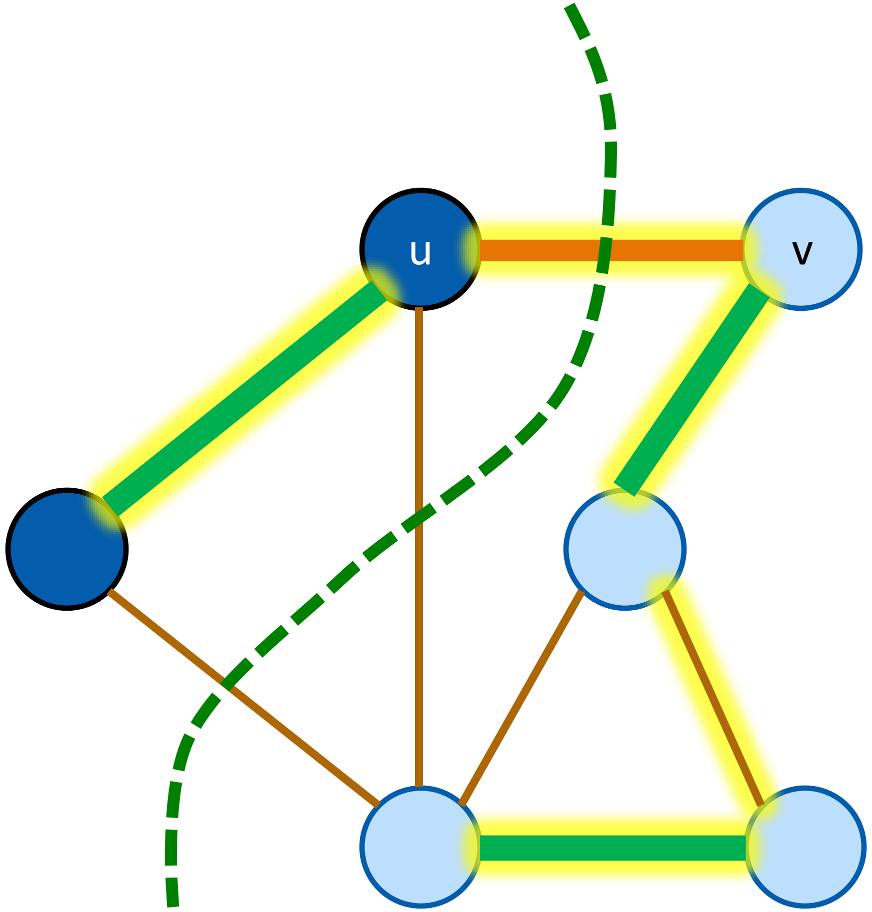
- Assume that we have:
  - a **cut** that respects **S**
  - **S** is part of some MST T.



Brief Aside – Cuts in Graphs

Proof of Lemma

- Assume that we have:
  - a **cut** that respects **S**
  - **S** is part of some MST T.
- Say that **{u,v}** is light.
  - lowest cost crossing the cut
- If **{u,v}** is in T, we are done.
  - T is an MST containing both {u,v} and S.



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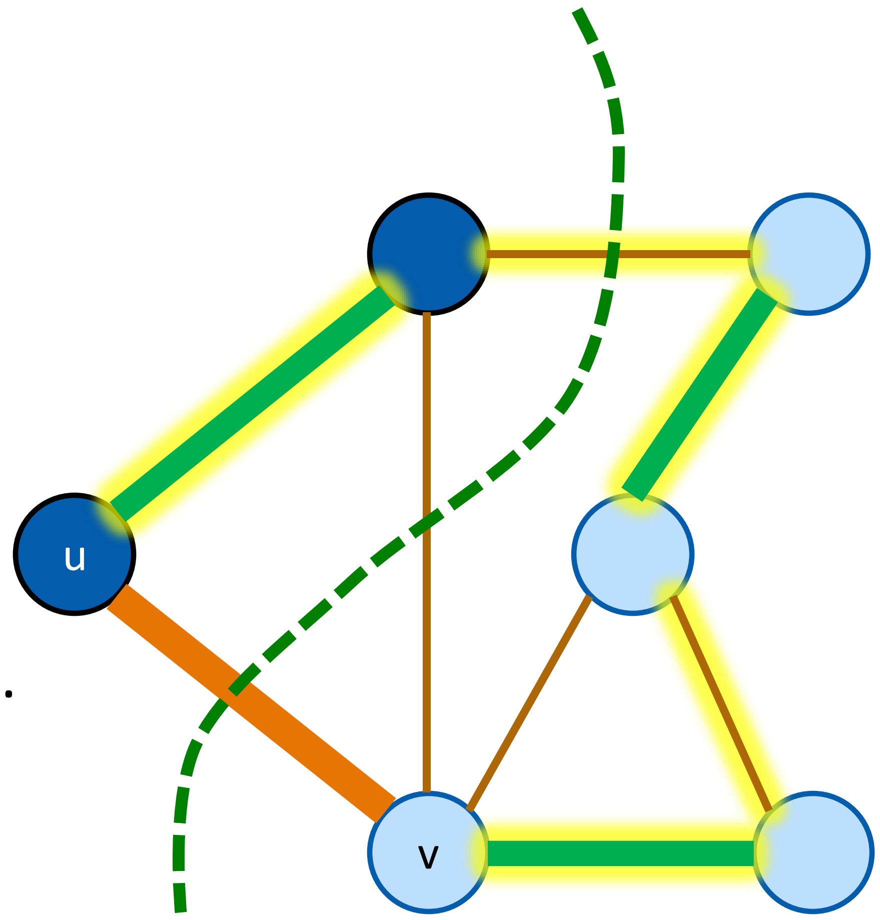
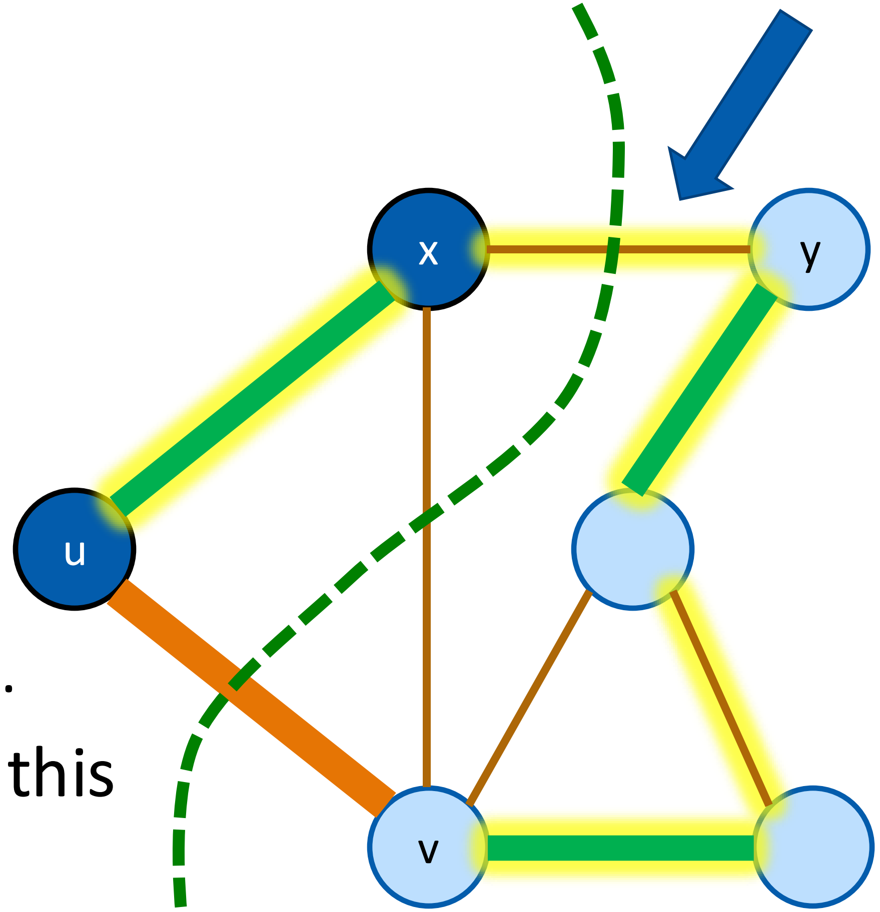
  - **S** is part of some MST T.
- Say that **{u,v}** is light.
  - lowest cost crossing the cut
- Say **{u,v}** is not in **T**.
  - Note that adding **{u,v}** to **T** will make a cycle.



Brief Aside – Cuts in Graphs

Proof of Lemma

- Assume that we have:
  - a **cut** that respects **S**
  - **S** is part of some MST T.
- Say that **{u,v}** is light.
  - lowest cost crossing the cut
- Say **{u,v}** is not in **T**.
  - Note that adding **{u,v}** to **T** will make a cycle.
- There is at least one other edge, **{x,y}**, in this cycle crossing the cut.

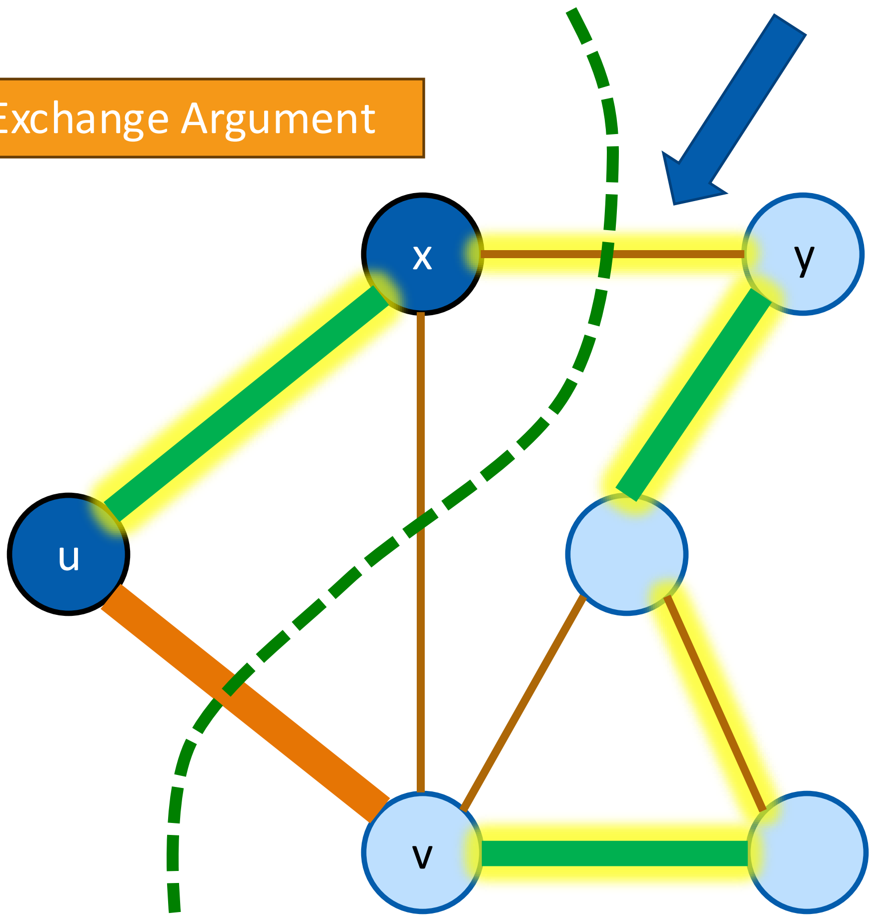


Brief Aside – Cuts in Graphs

Proof of Lemma ctd.

- Consider swapping {u,v} for {x,y} in **T**.
  - Call the resulting tree **T'**.

Exchange Argument

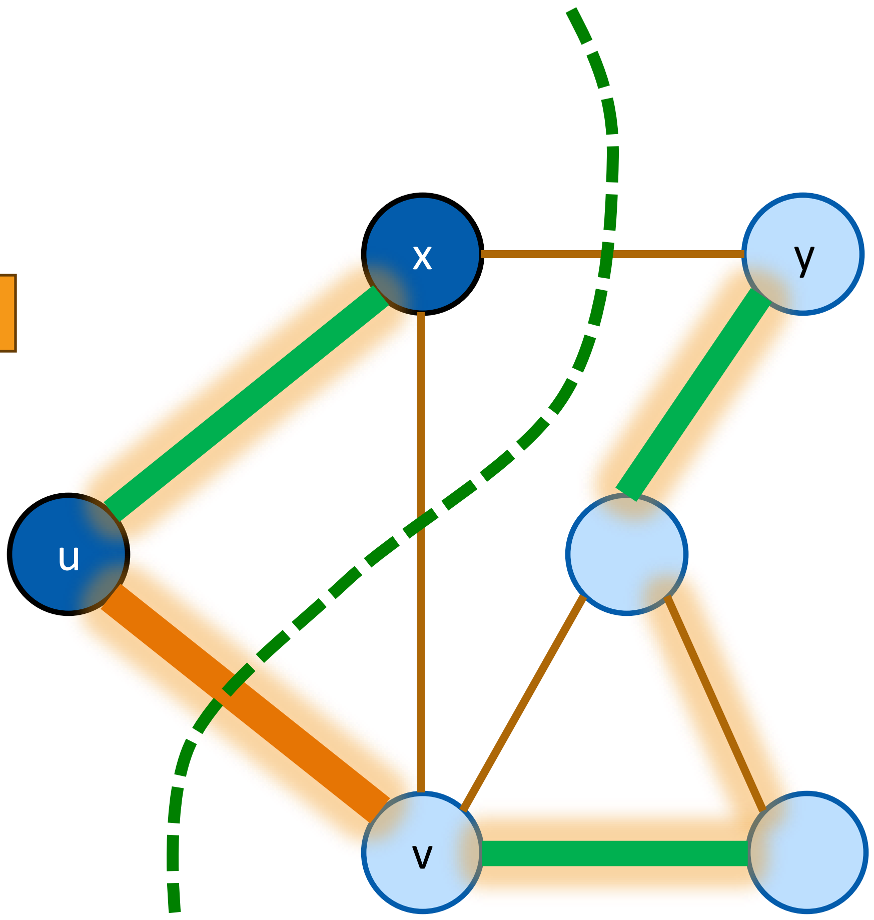


Brief Aside – Cuts in Graphs

Proof of Lemma ctd.

- Consider swapping {u,v} for {x,y} in **T**.
  - Call the resulting tree **T'**.
- **Claim**: **T'** is still an MST.
  - It is still a spanning tree (why?)
  - It has cost at most that of **T**
  - **T** had minimal cost.
  - So **T'** does too.
- So **T'** is an MST containing S and {u,v}.

Verification (easy)



How to find an MST

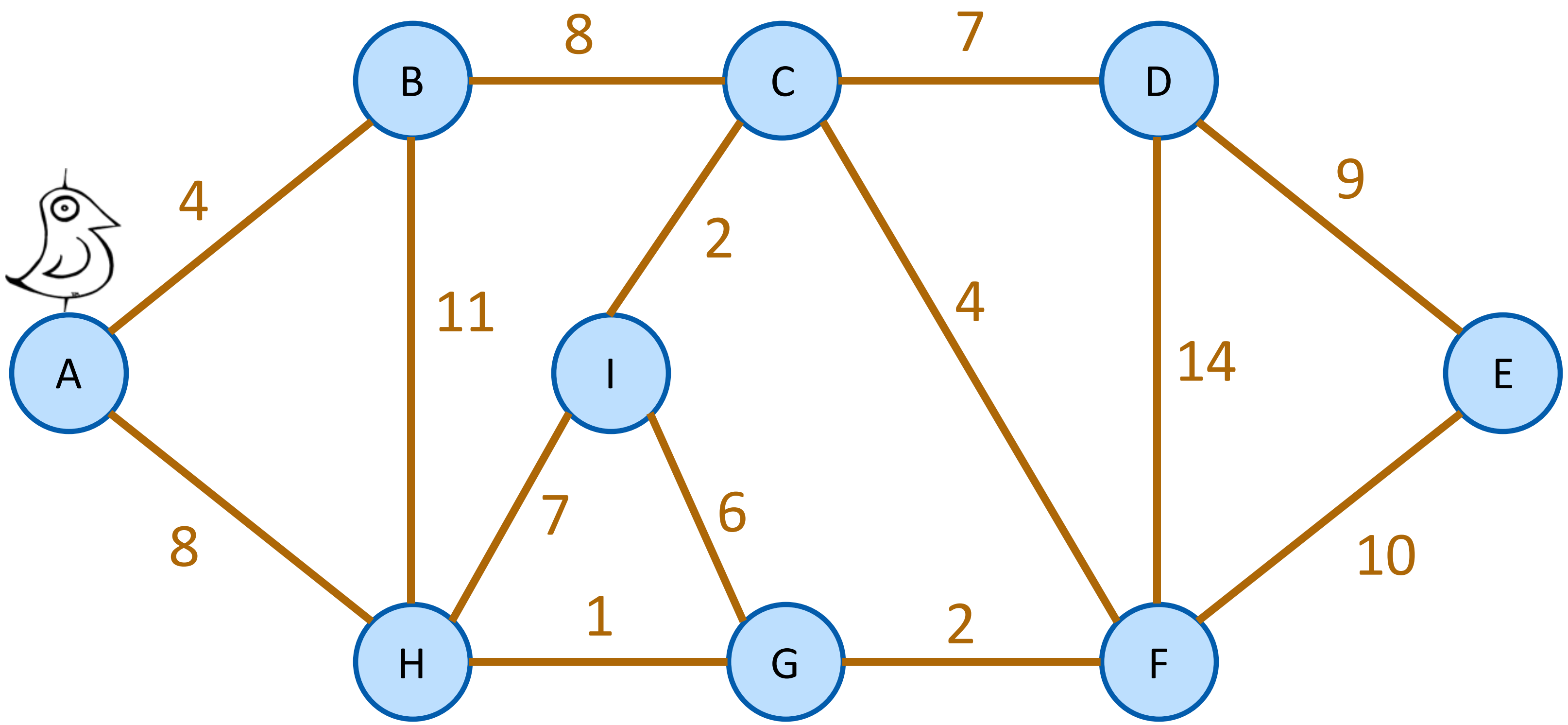
- How do we find one?
- Today we'll see **two greedy algorithms**.

- The strategy:
  - Make a **series of choices,** adding edges to the tree.
  - Show that each edge we add is **safe to add**:
    - we do not rule out the possibility of success
    - we will choose **light edges** crossing **cuts** and **use the Lemma**.
  - **Keep going** until we have an MST.

How to find an MST

Idea:

Start growing a tree, greedily add the shortest edge we can to grow the tree.



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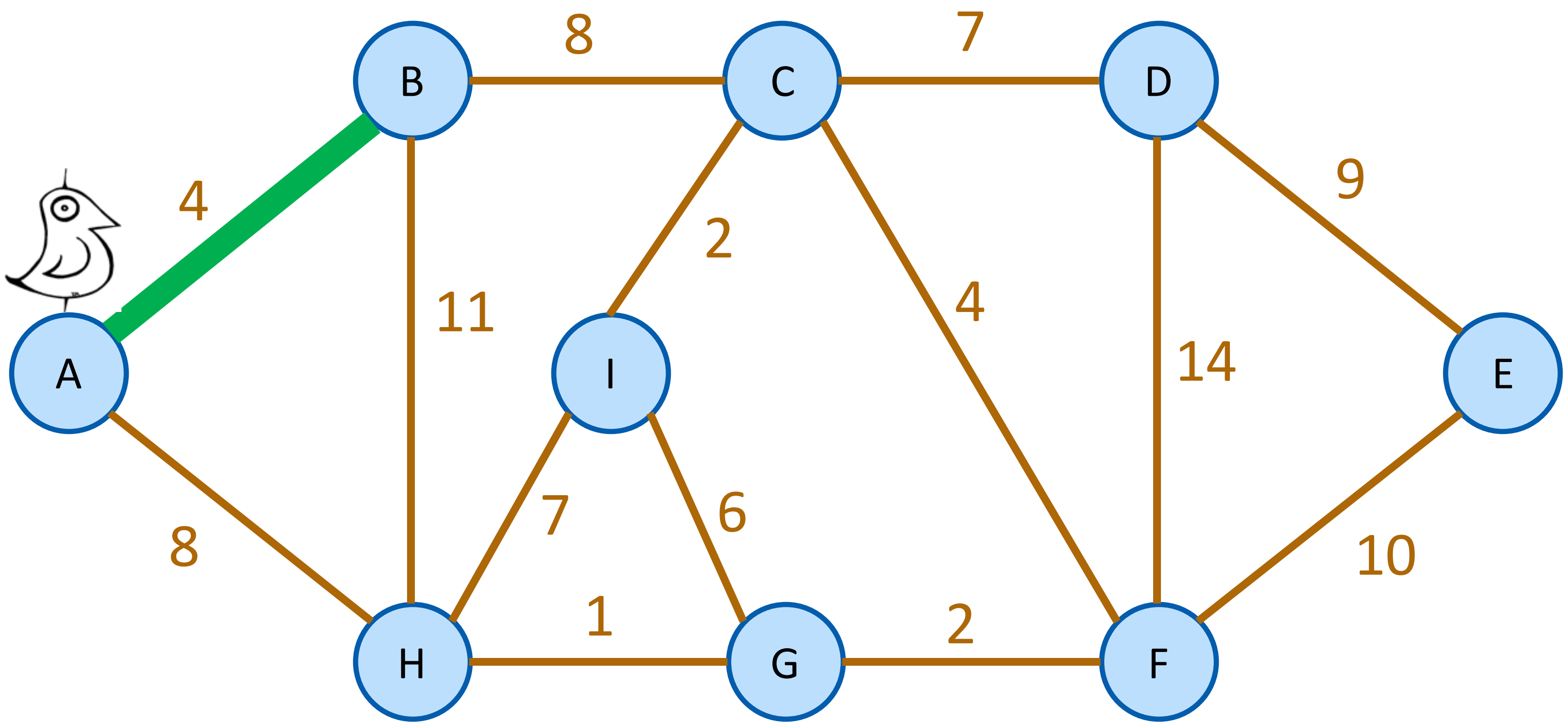

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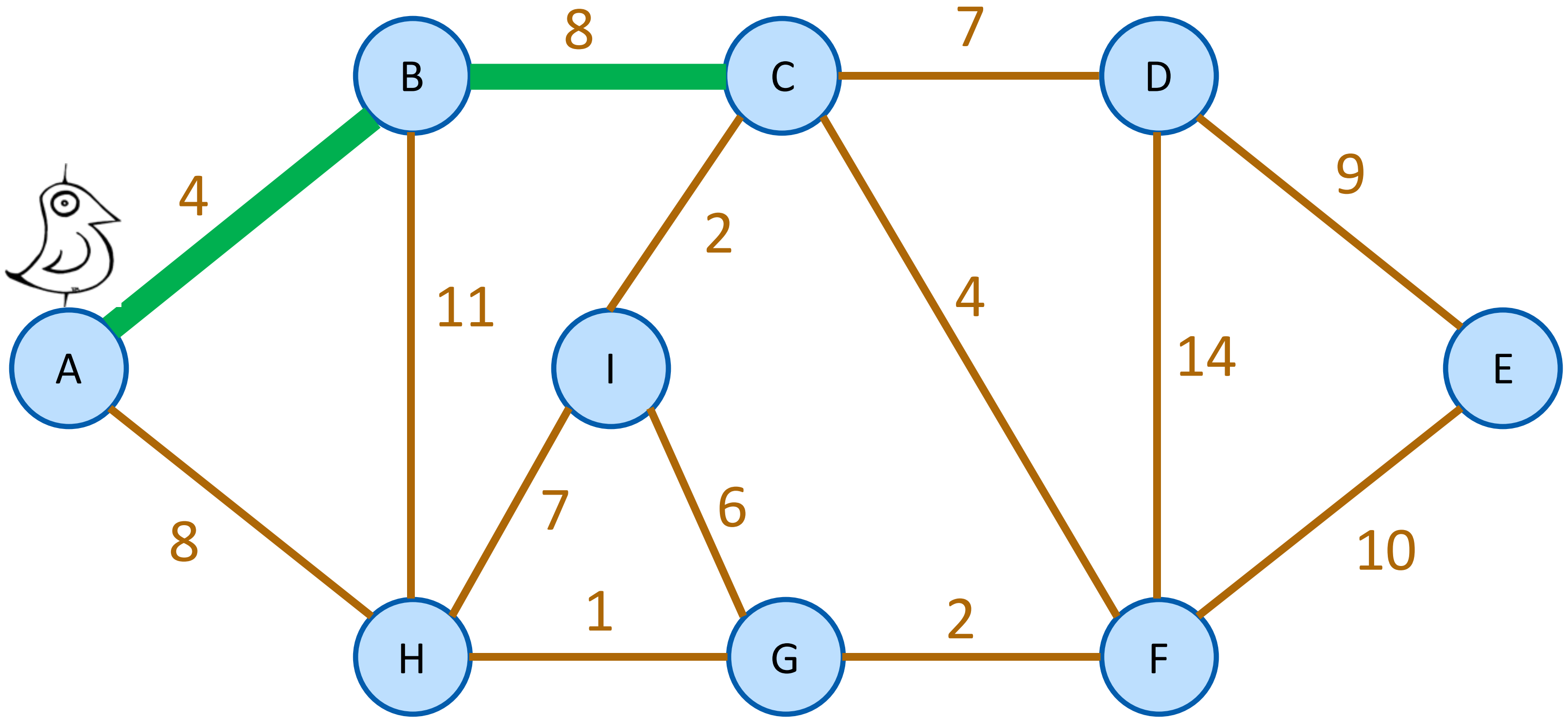

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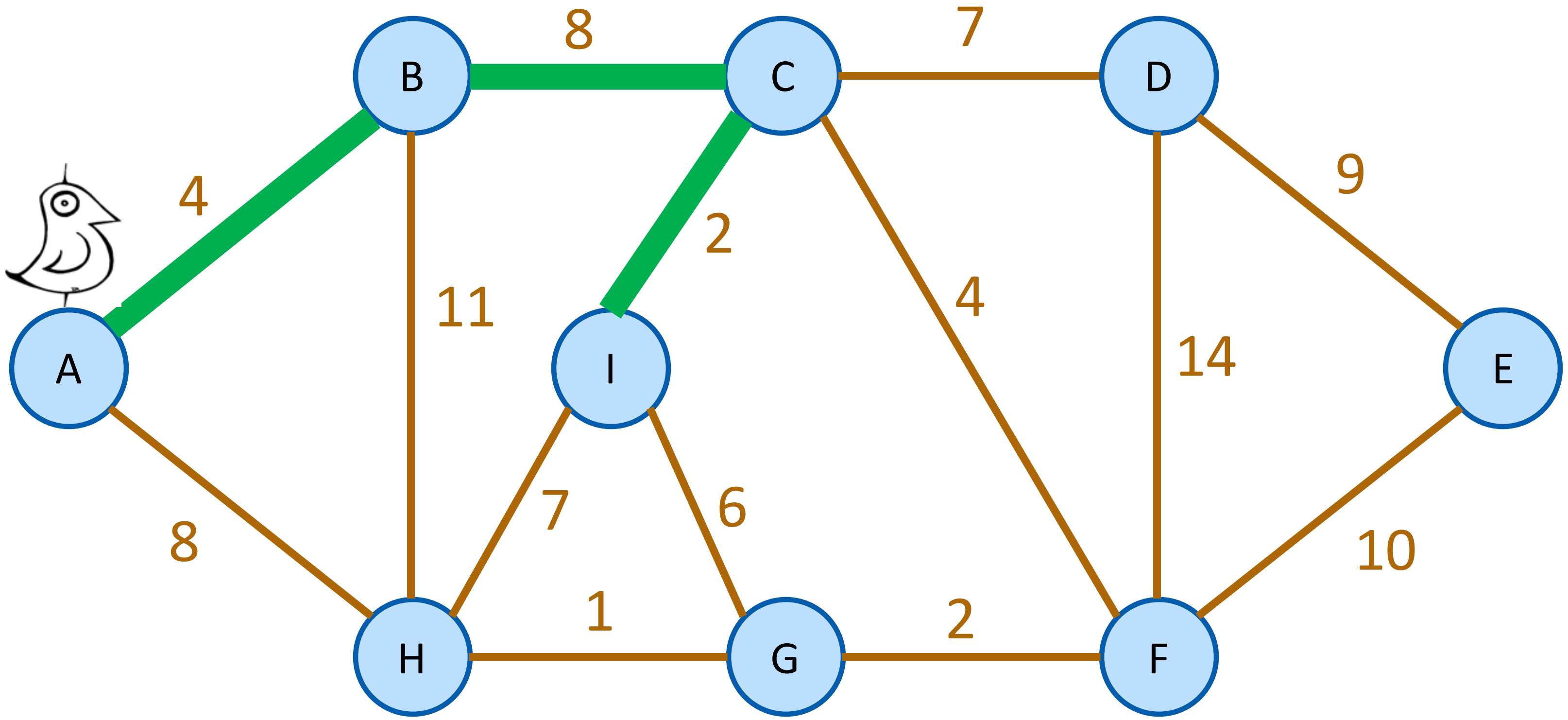

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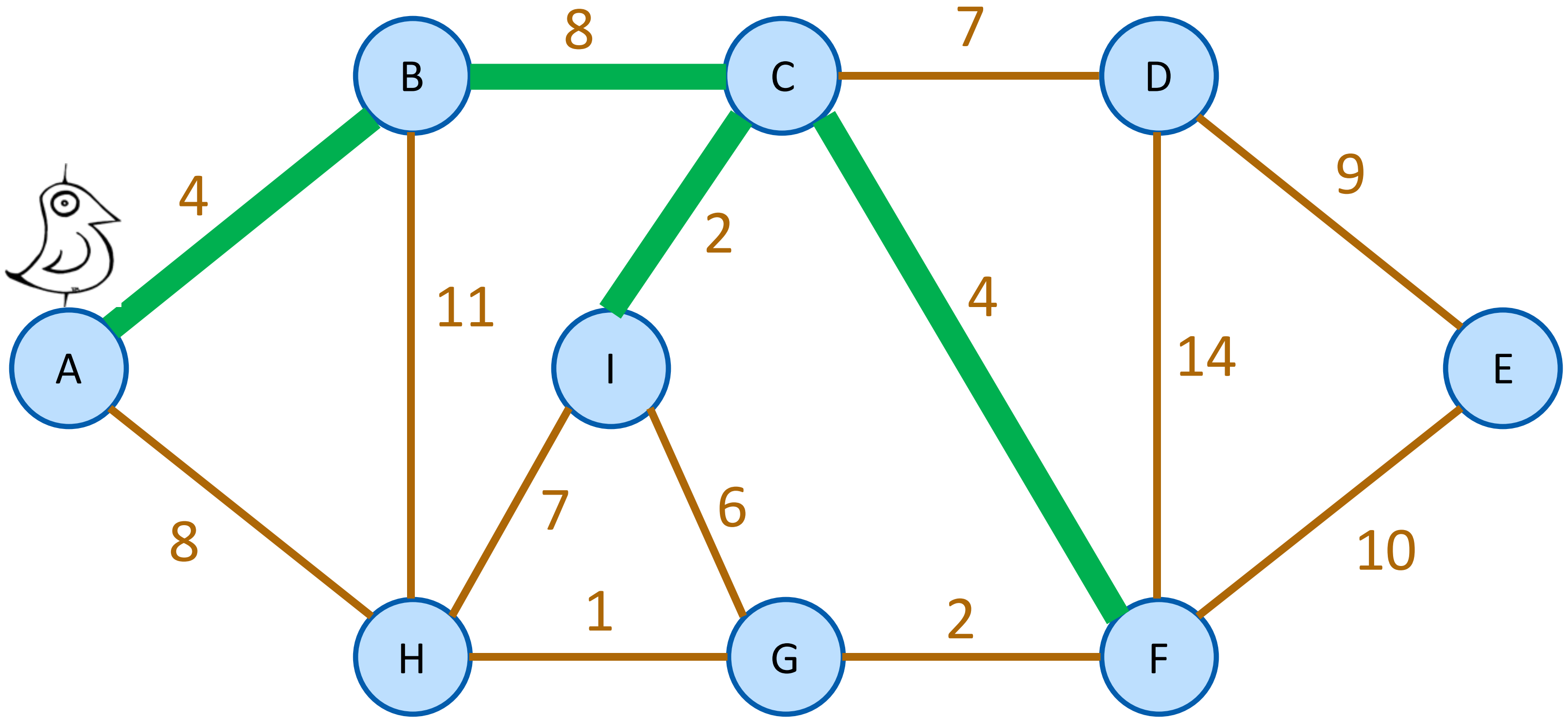

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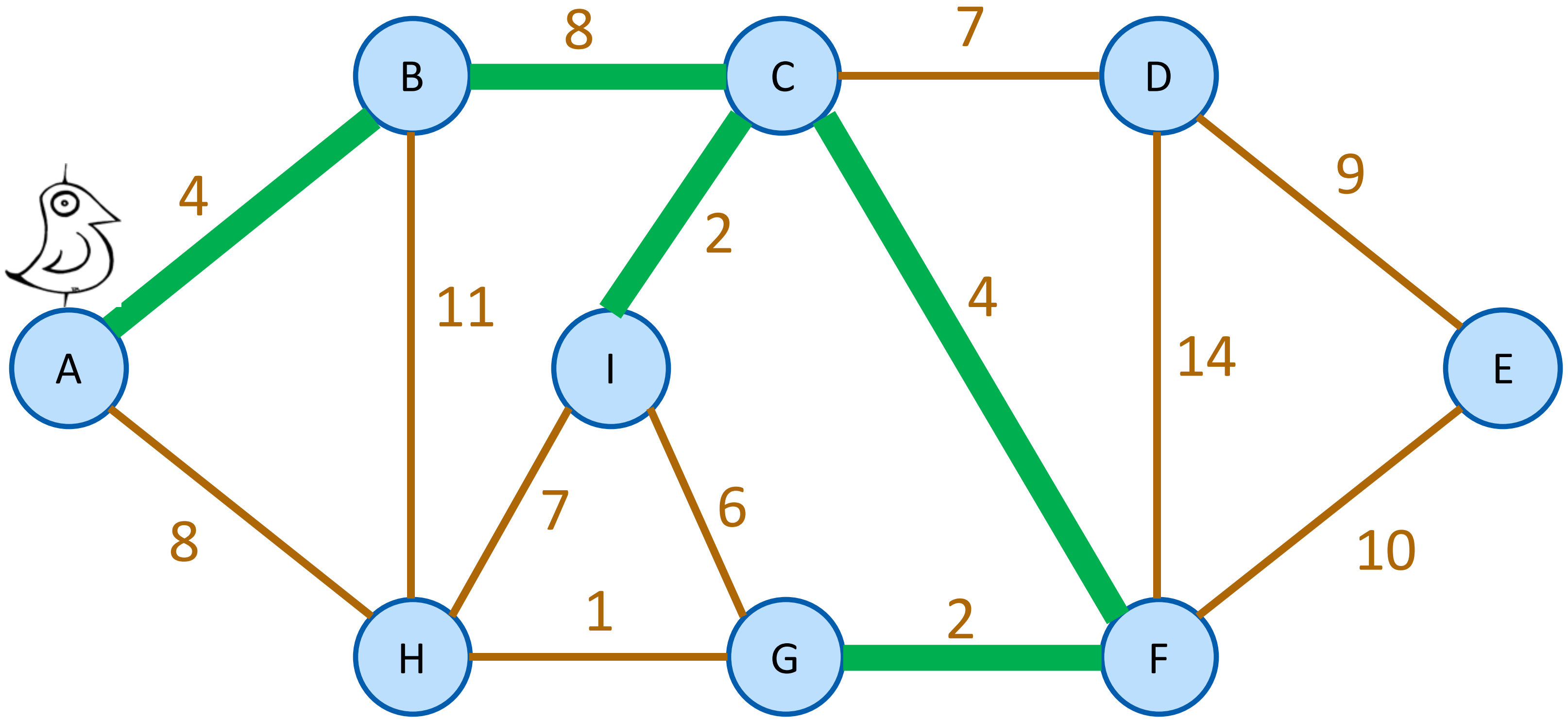

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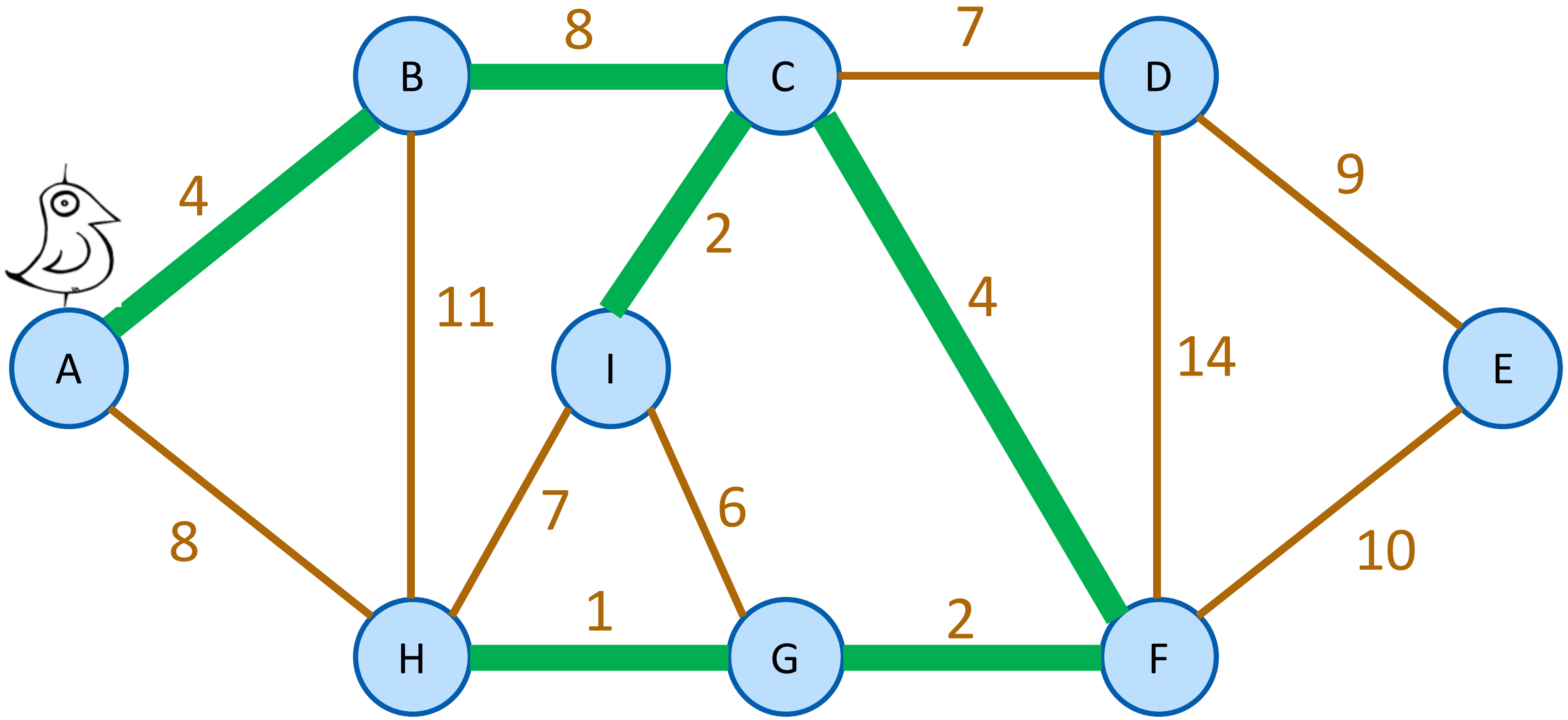

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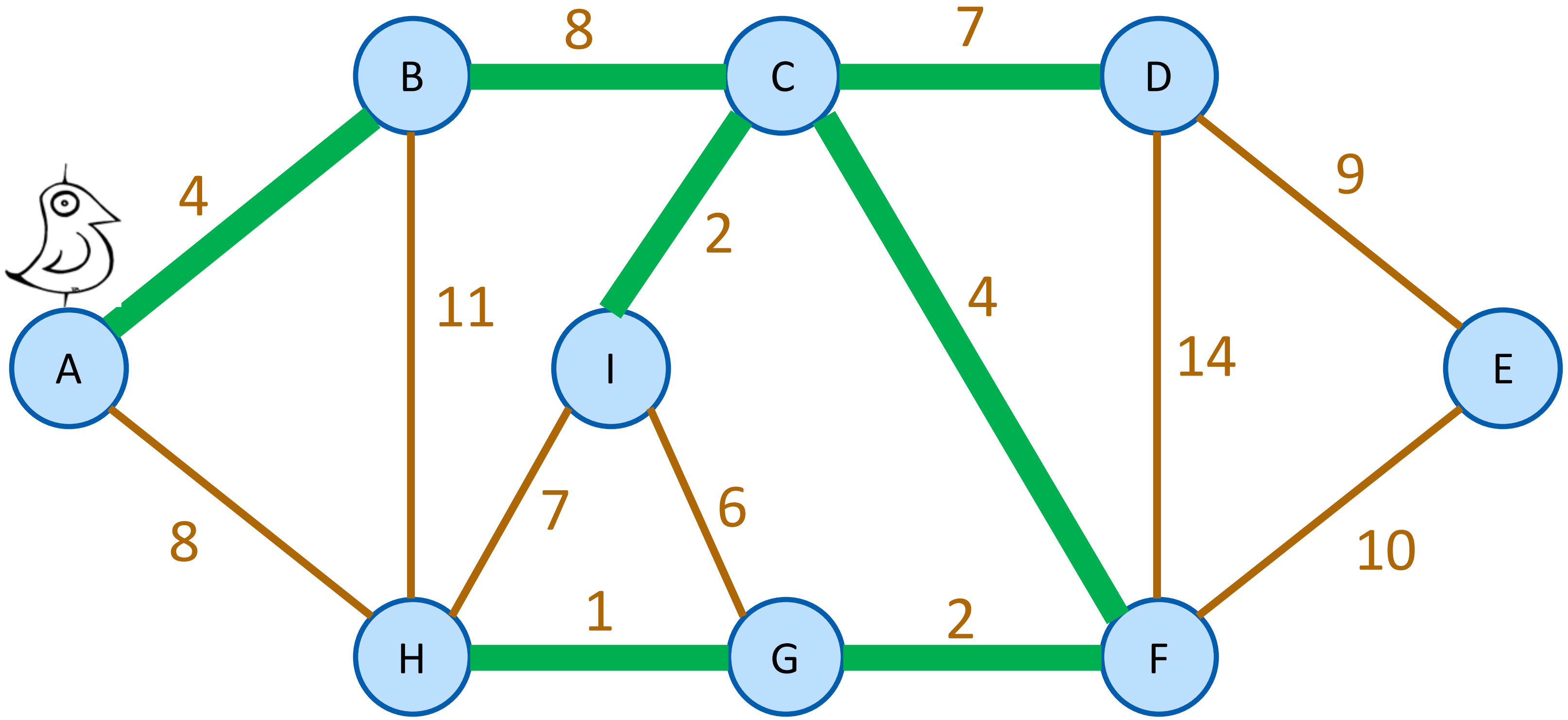

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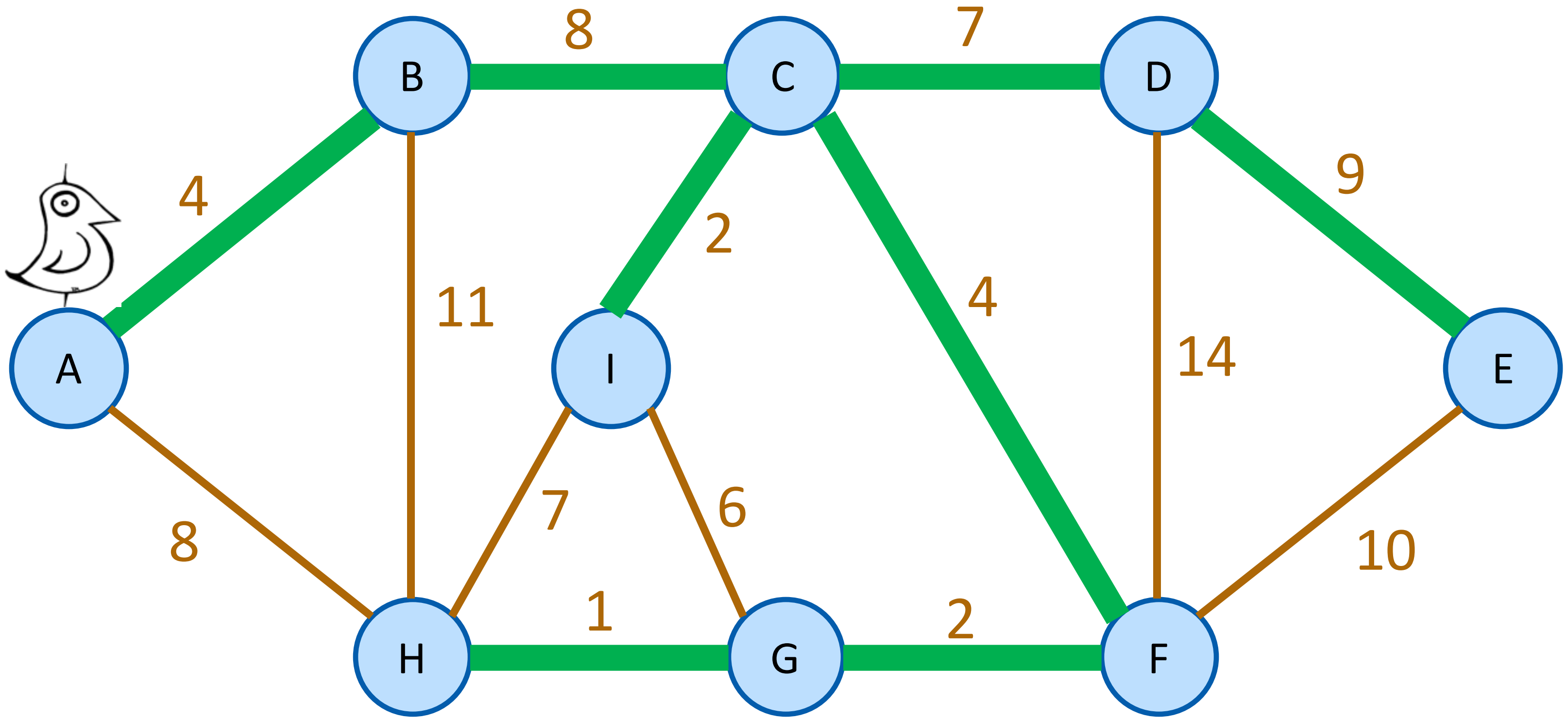

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We've discovered Prim's algorithm!

- slowPrim(G = (V,E), starting vertex s):
  - MST = {}
  - verticesVisited = { s }
  - **while** |verticesVisited| < |V|:
    - find the lightest edge {x,v} in E so that:
      - x is in verticesVisited
      - v is not in verticesVisited
    - add {x,v} to MST
    - add v to verticesVisited
  - **return** MST

Naively, the running time is O(nm):

- For each of ≤n-1 iterations of the while loop:
  - Go through all the edges.

Prim's Algorithm

Two questions

1. Does it work?
   - That is, does it actually return a MST?

2. How do we actually implement this?
   - the pseudocode above says "slowPrim"…

Prim's Algorithm

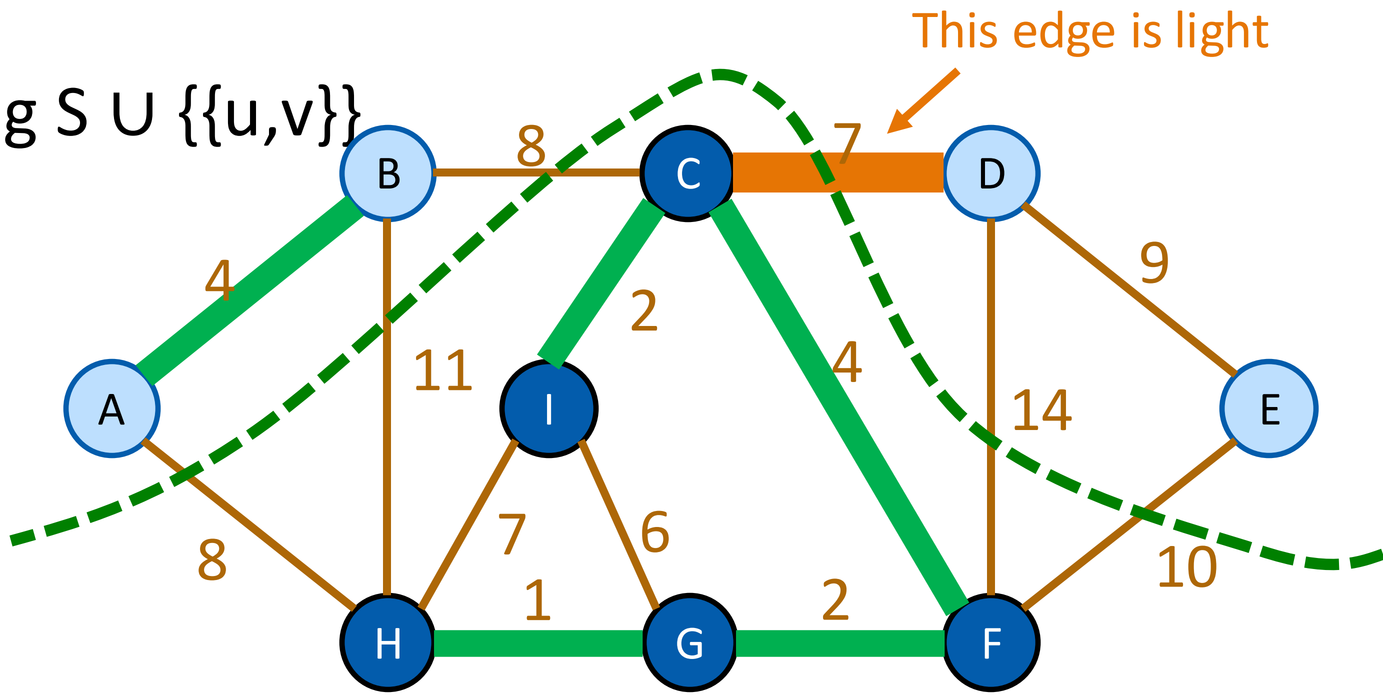
Does it work?

- We need to show that our greedy choices **don't rule out success.**
- That is, at every step:
  - If there exists an MST that contains all of the edges S we have added so far…
  - …then when we make our next choice {u,v}, there is still an MST containing S and {u,v}.

- Now it is time to use our lemma!

Lemma

- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
- Let {u,v} be a light edge.
- Then there is an MST containing S U {{u,v}}

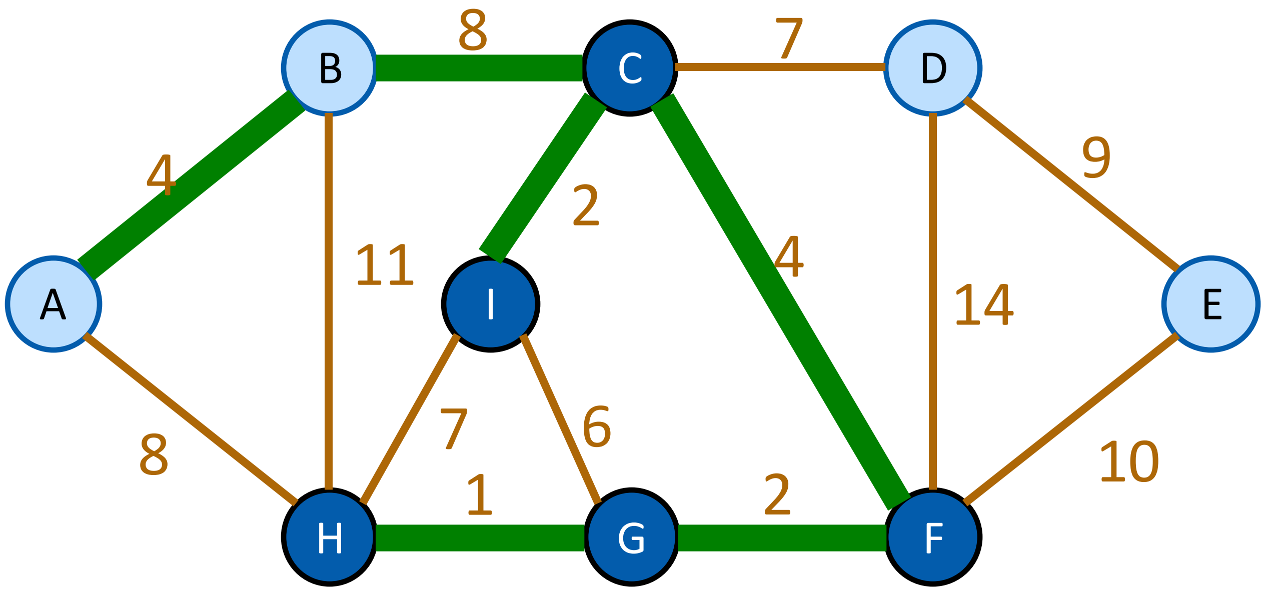


Prim's Algorithm

- Assume that our choices **S** so far don't rule out success
  - There is an MST consistent with those choices

How can we use our lemma to show that our next choice also does not rule out success?

S is the set of edges selected so far



Prim's Algorithm

- Assume that our choices **S** so far don't rule out success
  - There is an MST consistent with those choices
- Consider the cut {**visited**, **unvisited**}
  - This cut respects S.

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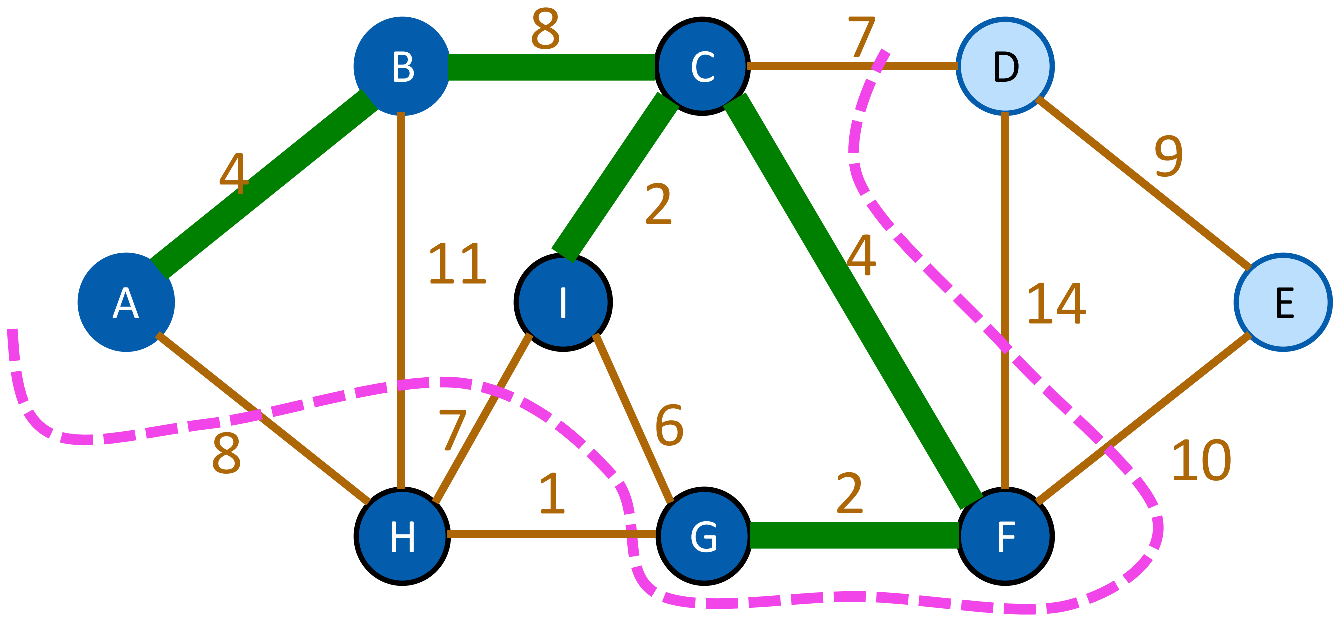
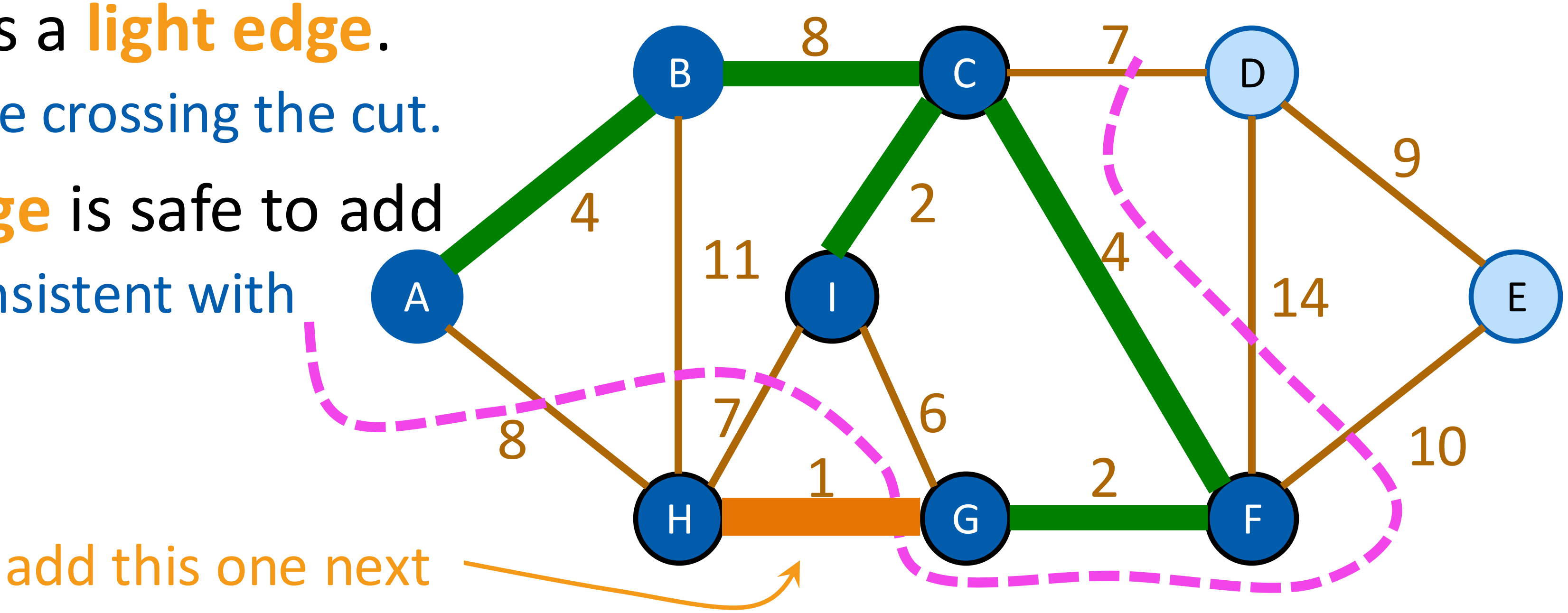


Prim's Algorithm

- Assume that our choices **S** so far don't rule out success
  - There is an MST consistent with those choices
- Consider the cut {**visited**, **unvisited**}
  - This cut respects S.
- The edge we add next is a **light edge**.
  - Least weight of any edge crossing the cut.
- By the Lemma, **that edge** is safe to add
  - There is still an MST consistent with the new set of edges.

S is the set of edges selected so far



Prim's Algorithm

Formally,

- Inductive hypothesis:
  - After adding the t'th edge, there exists an MST with the edges added so far.
- Base case:
  - In the beginning, with no edges added, there exists an MST containing all the (zero) edges added so far. **YEP.**
- Inductive step:
  - If the inductive hypothesis holds for t (aka, the choices so far are safe), then it holds for t+1 (aka, the next edge we add is safe).
  - **That's what we just showed.**
- Conclusion:
  - After adding the n-1'st edge, there exists an MST with the edges added so far.
  - At this point, we have a spanning tree, so it better be a minimum spanning tree.

Prim's Algorithm

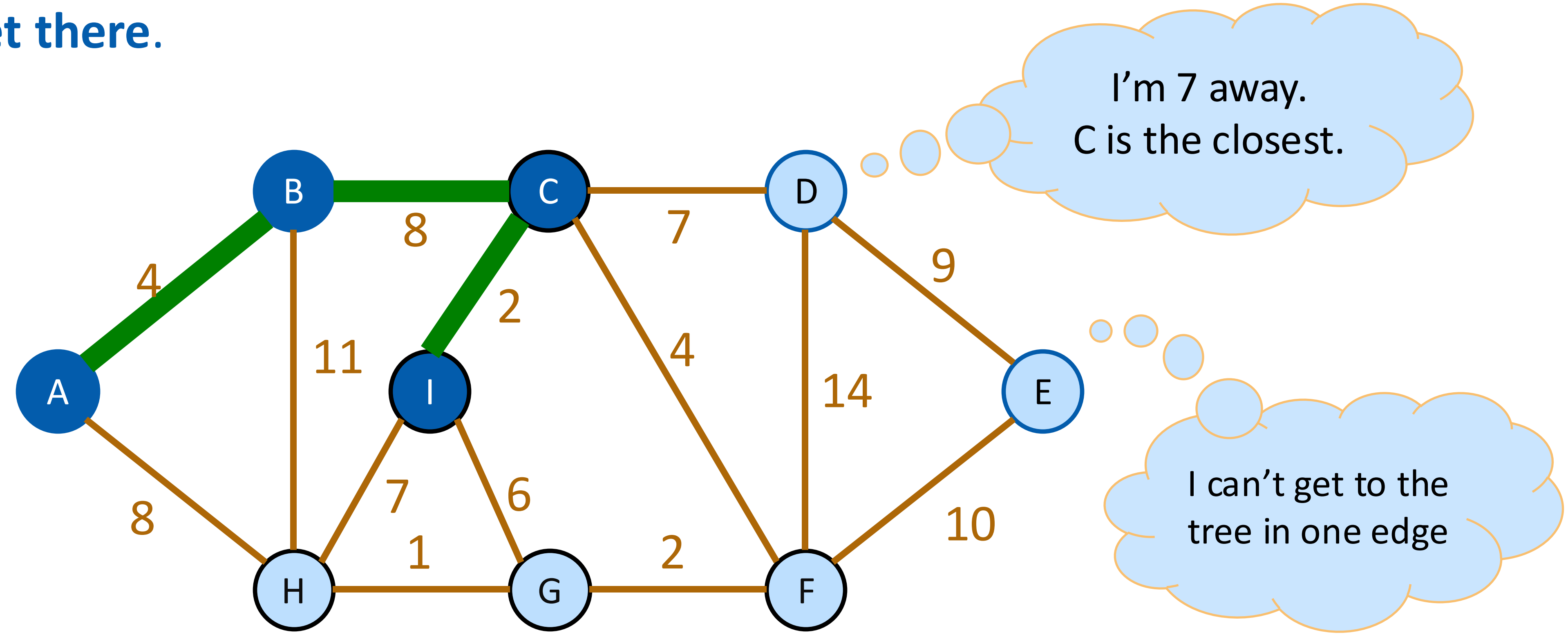
Two questions

1. Does it work?
   - That is, does it actually return a MST?
     - **YES!**

2. How do we actually implement this?
   - the pseudocode above says "slowPrim"...

Efficient Implementation

- Each vertex keeps:
  - the **(single-edge) distance** from itself to the **growing spanning tree**
  - **how to get there**.



Efficient Implementation

- Each vertex keeps:
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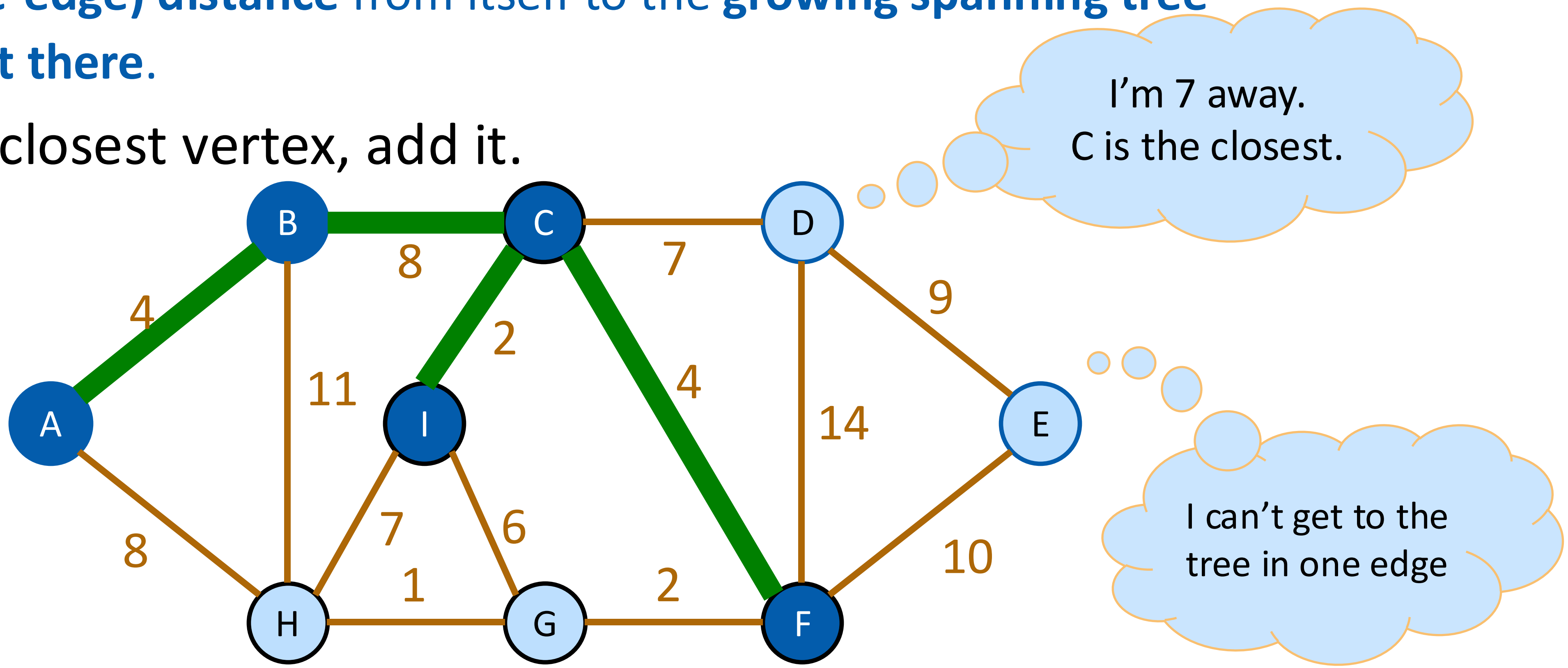
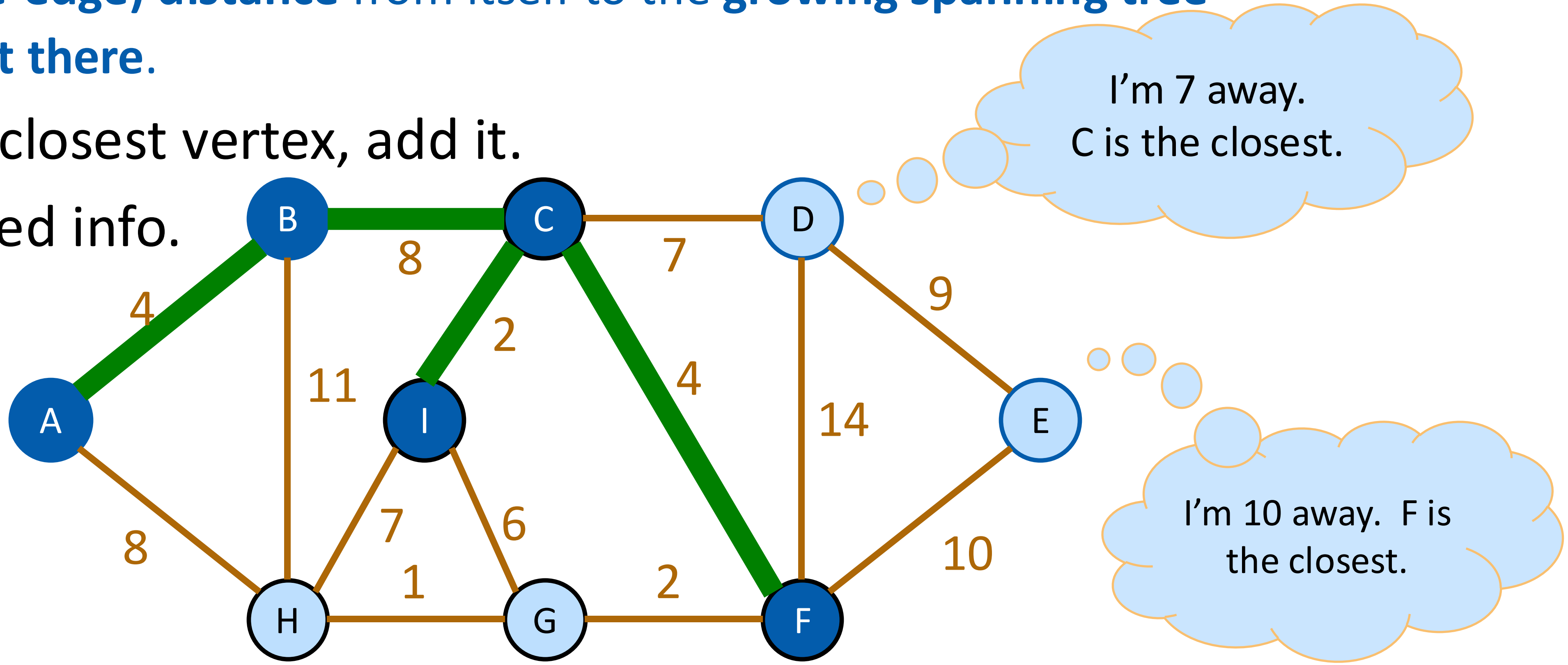
- Choose the closest vertex, add it.



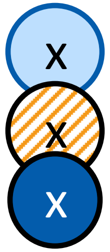
Efficient Implementation

- Each vertex keeps:
  - the **(single-edge) distance** from itself to the **growing spanning tree**
  - **how to get there**.
- Choose the closest vertex, add it.
- Update stored info.



Efficient Implementation

Every vertex has a key and a parent

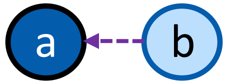


Can't reach x yet

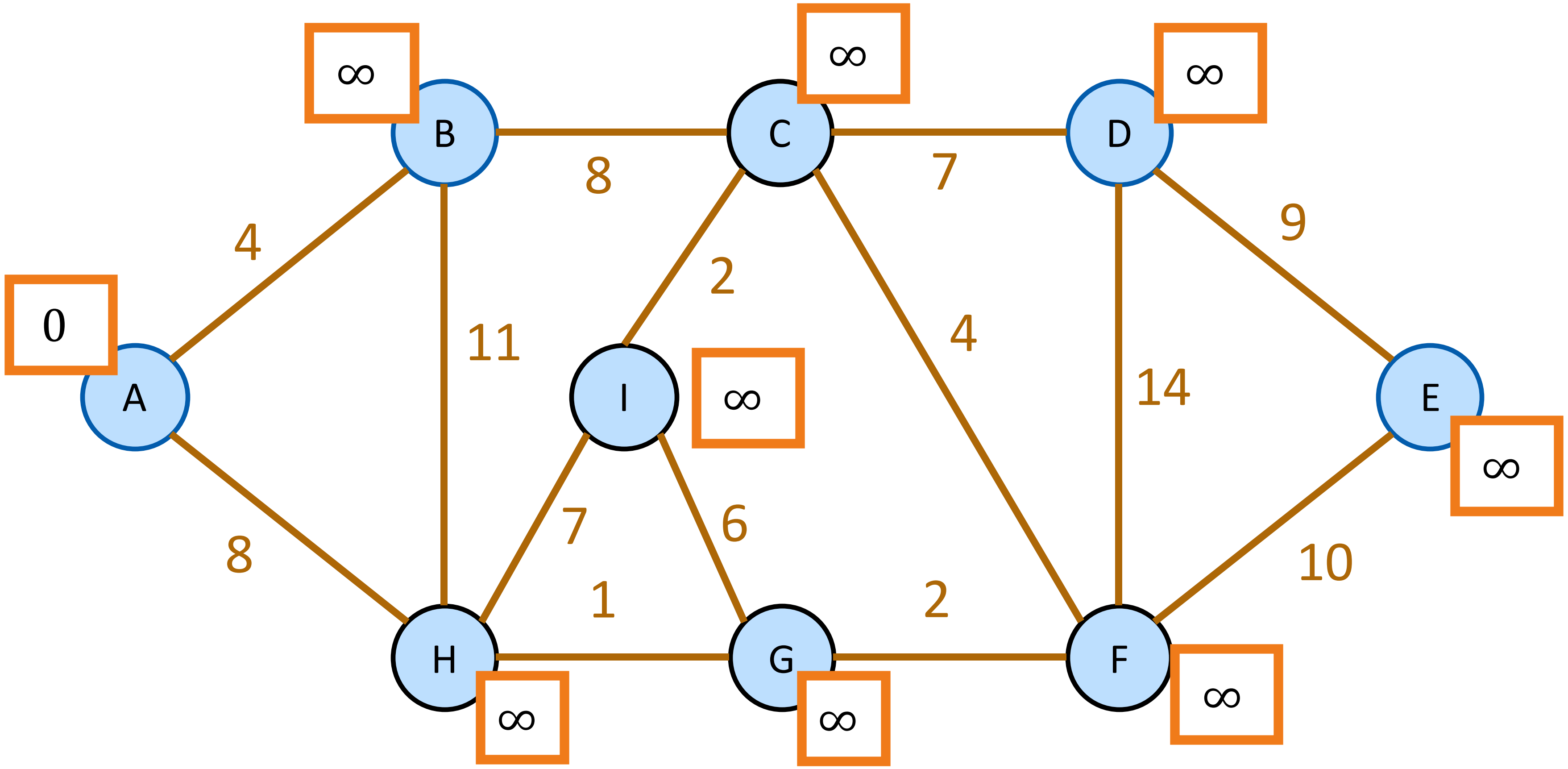
x is "active"

Can reach x

$k[x]$ k[x] is the distance of x from the growing tree



p[b] = a, meaning that a was the vertex that k[b] comes from.



Until all the vertices are **reached:**

- Activate the **unreached** vertex u with the smallest key.
- **for each** of u's unreached neighbors v:
  - k[v] = min(k[v], weight(u,v))
  - if k[v] updated, p[v] = u

Prim's Algorithm

Efficient Implementation

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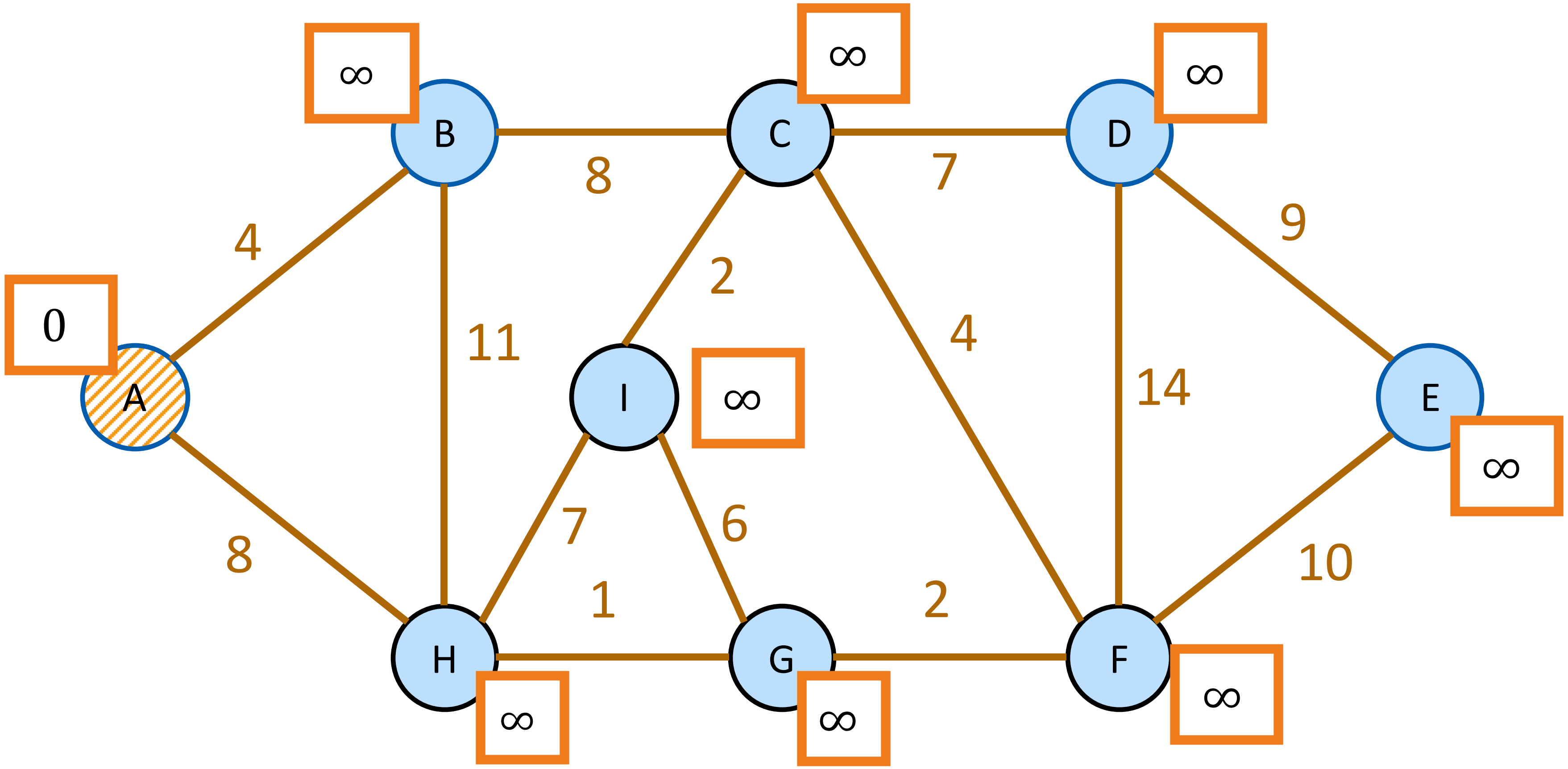

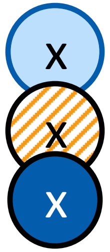

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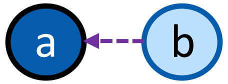

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Prim's Algorithm

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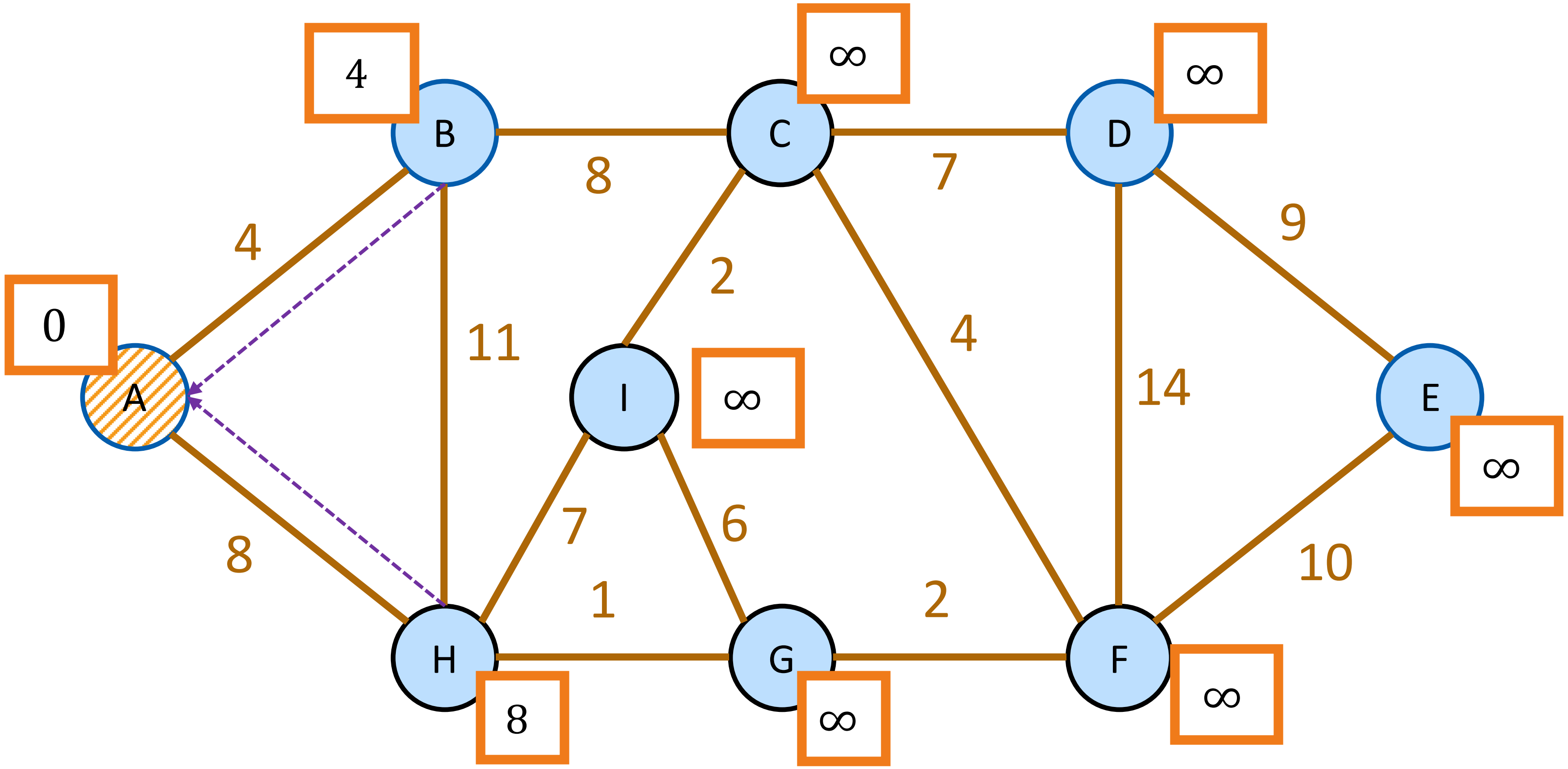

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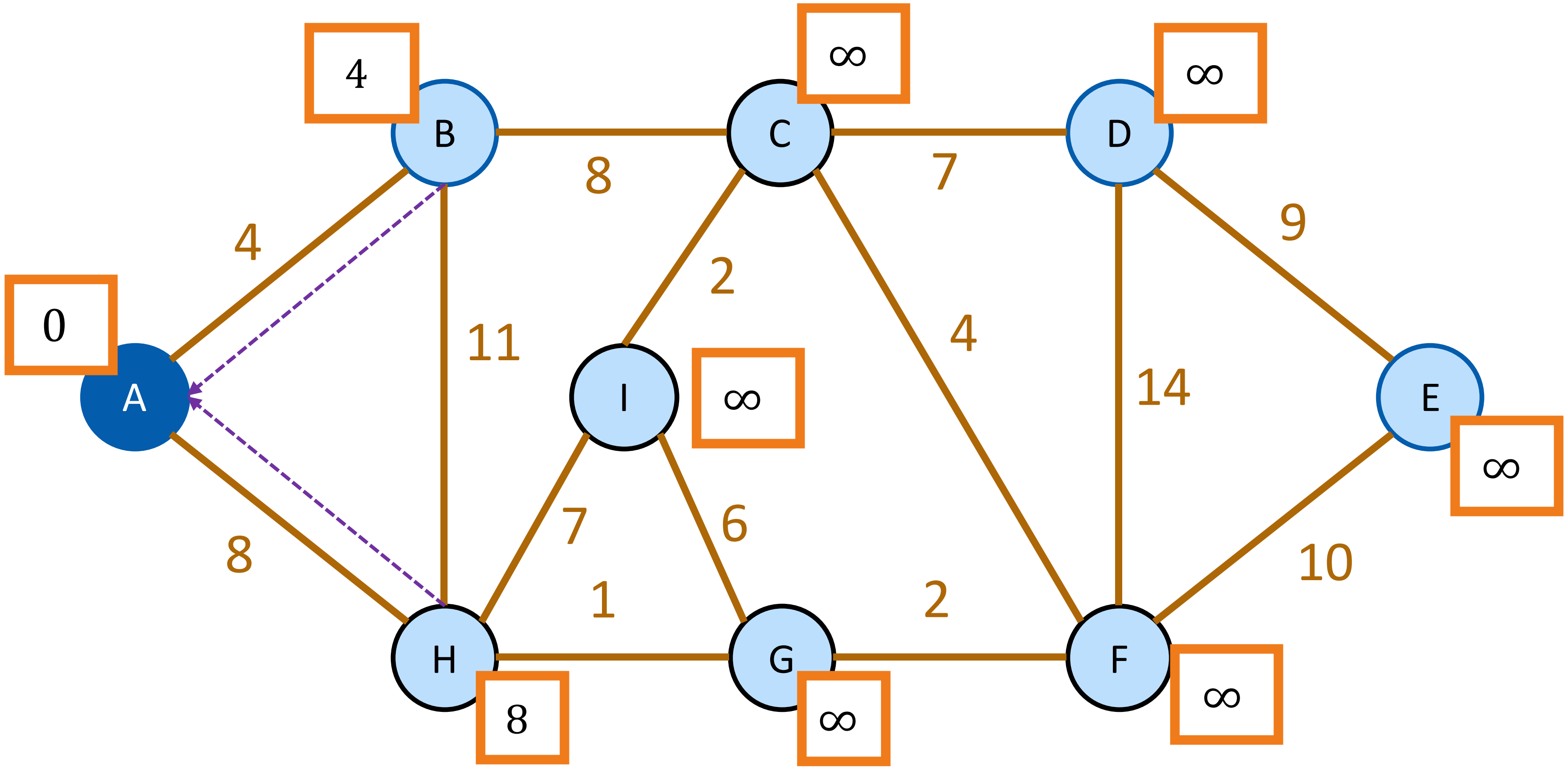

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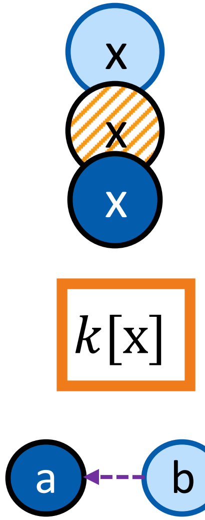

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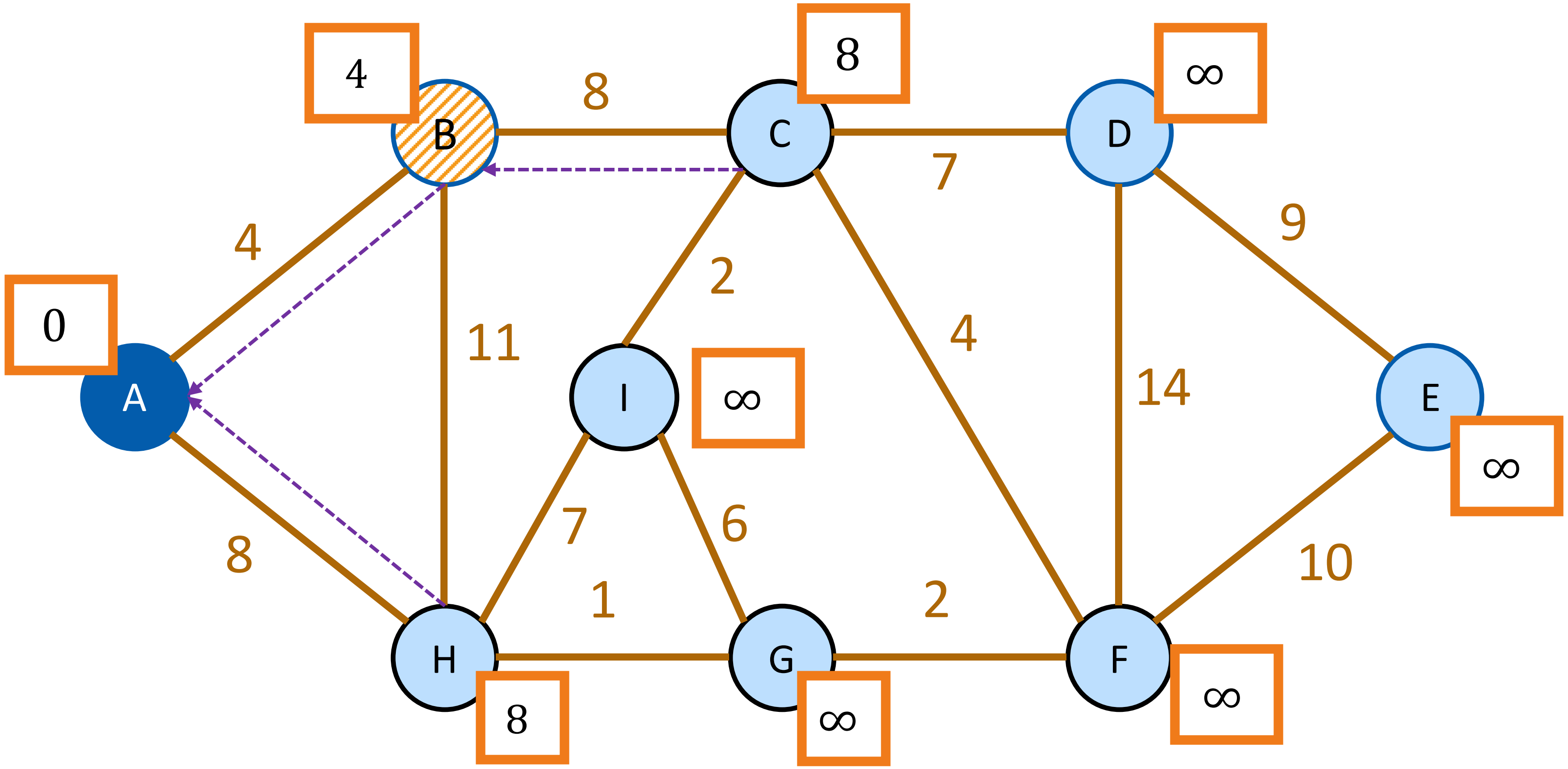

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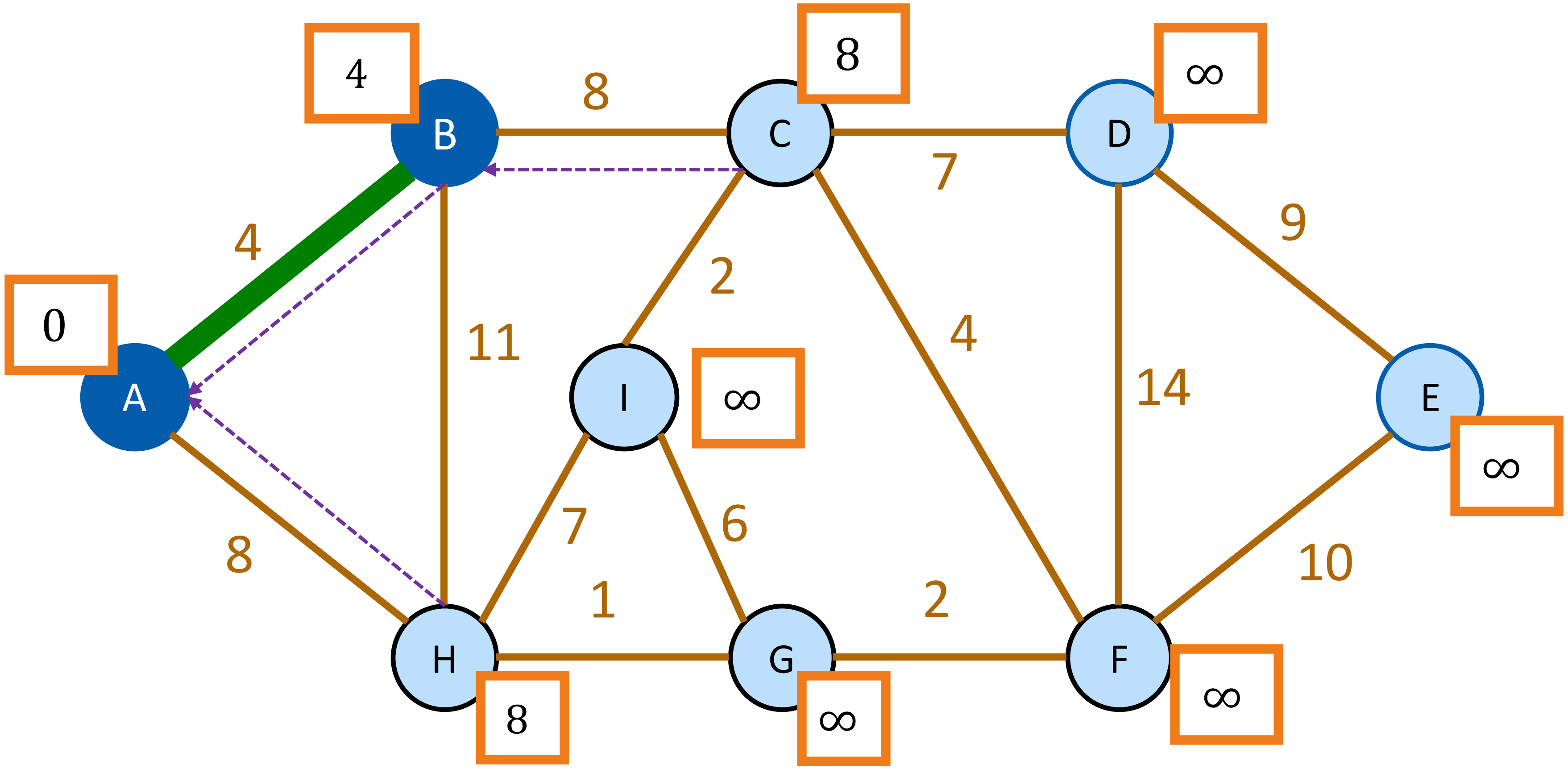

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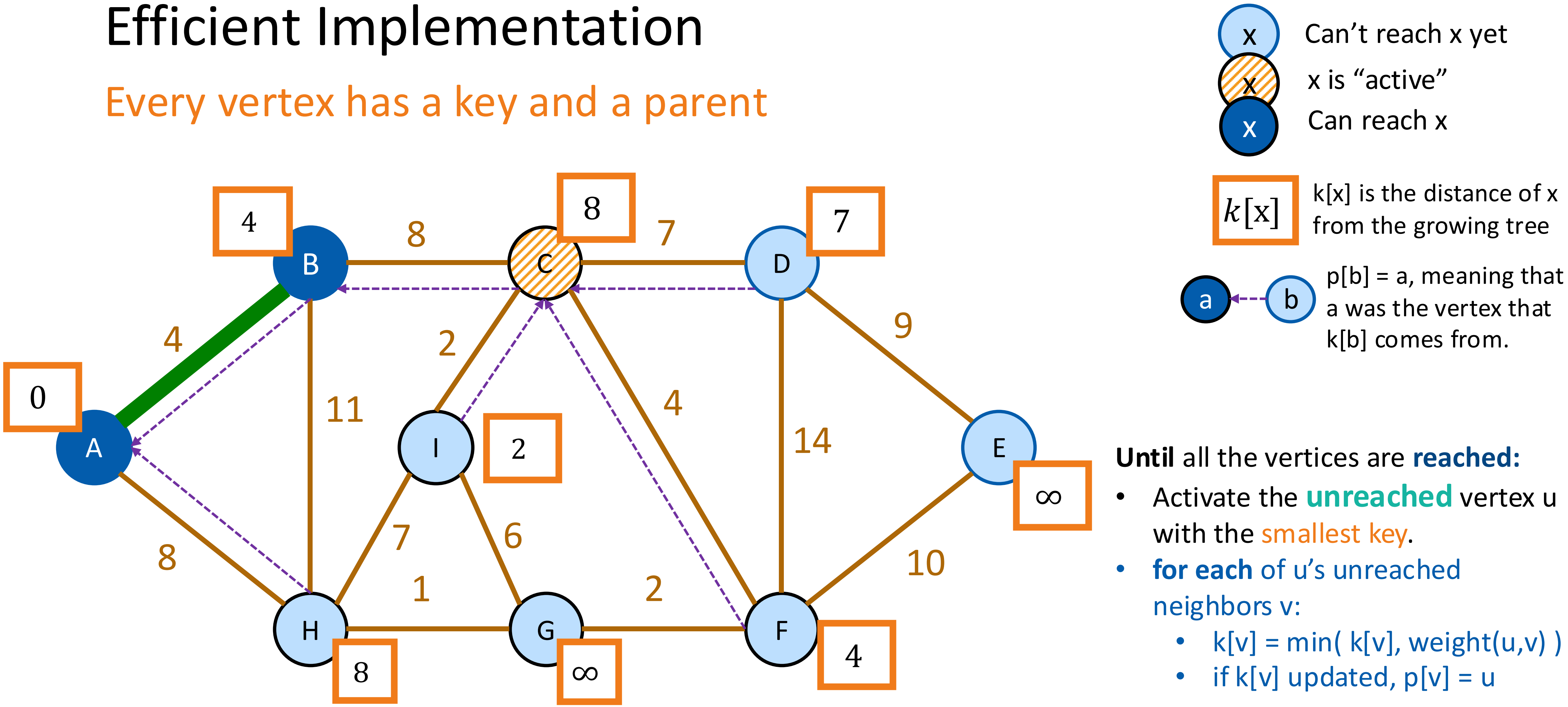

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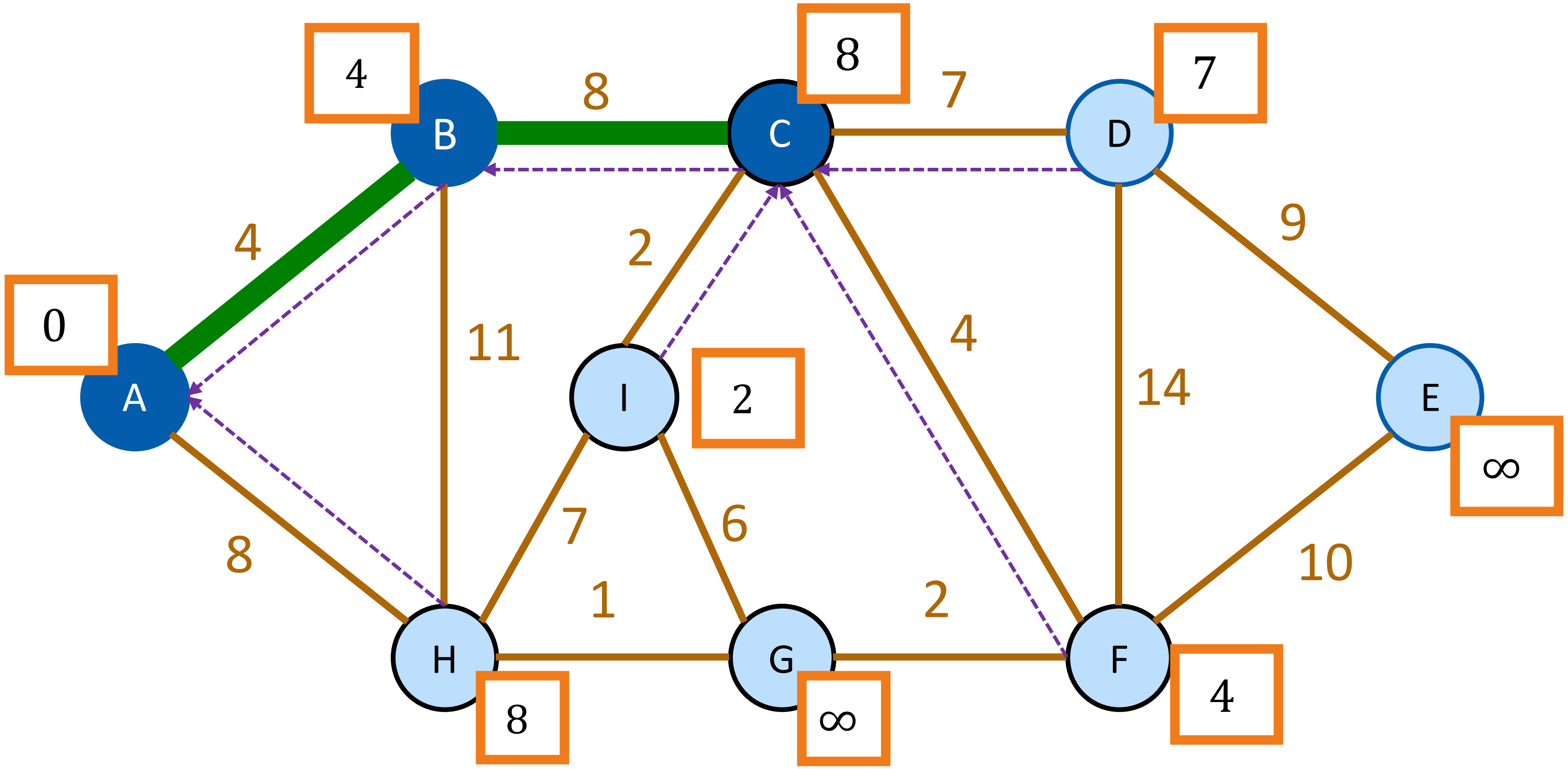

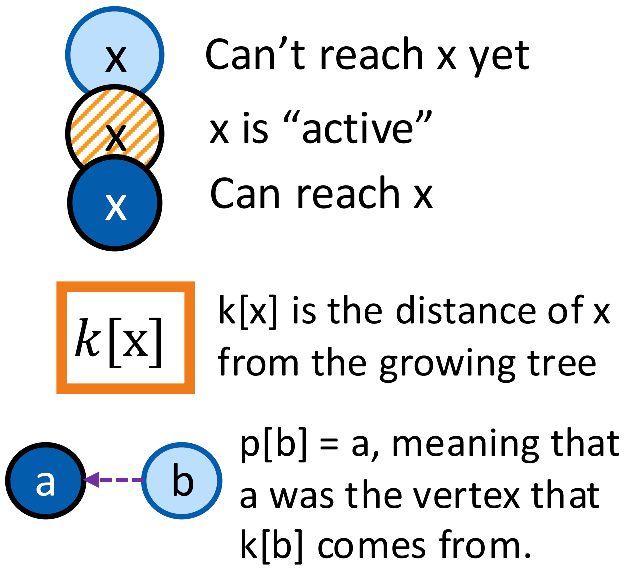

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Prim’s Algorithm

Efficient Implementation

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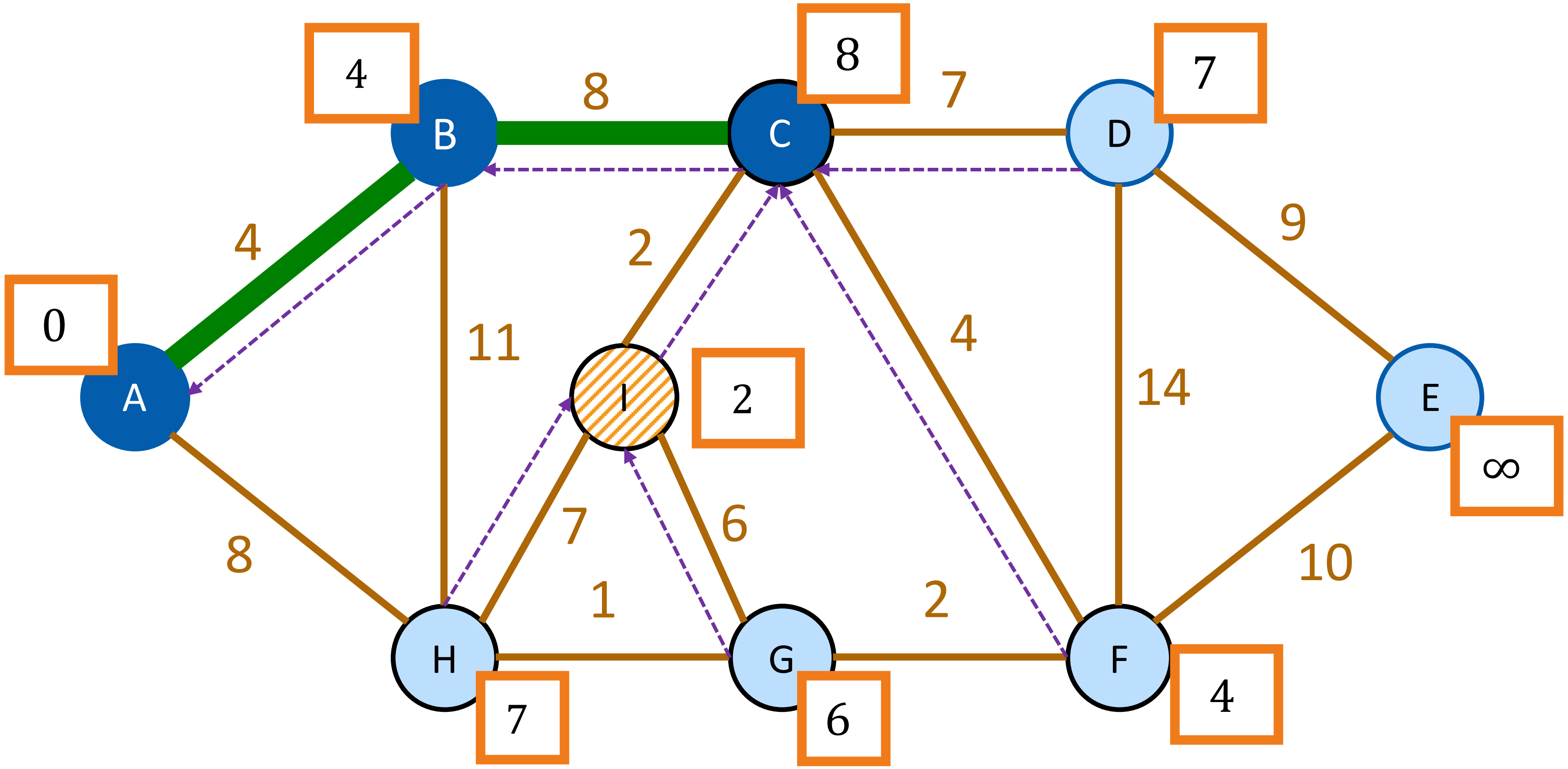

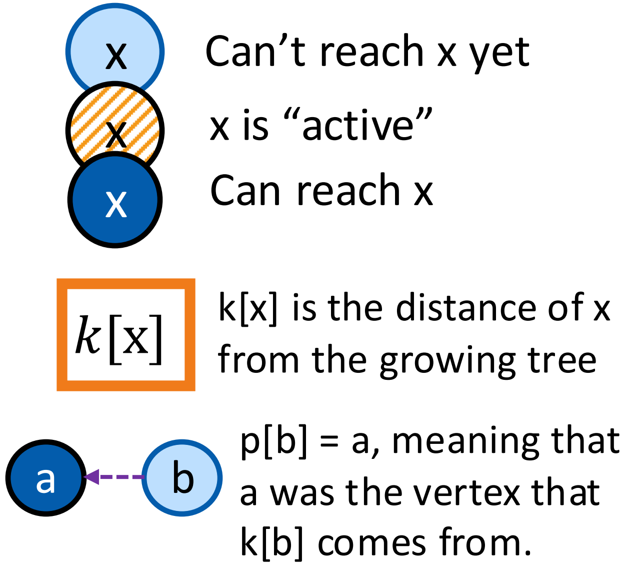

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Prim's Algorithm

Efficient Implementation

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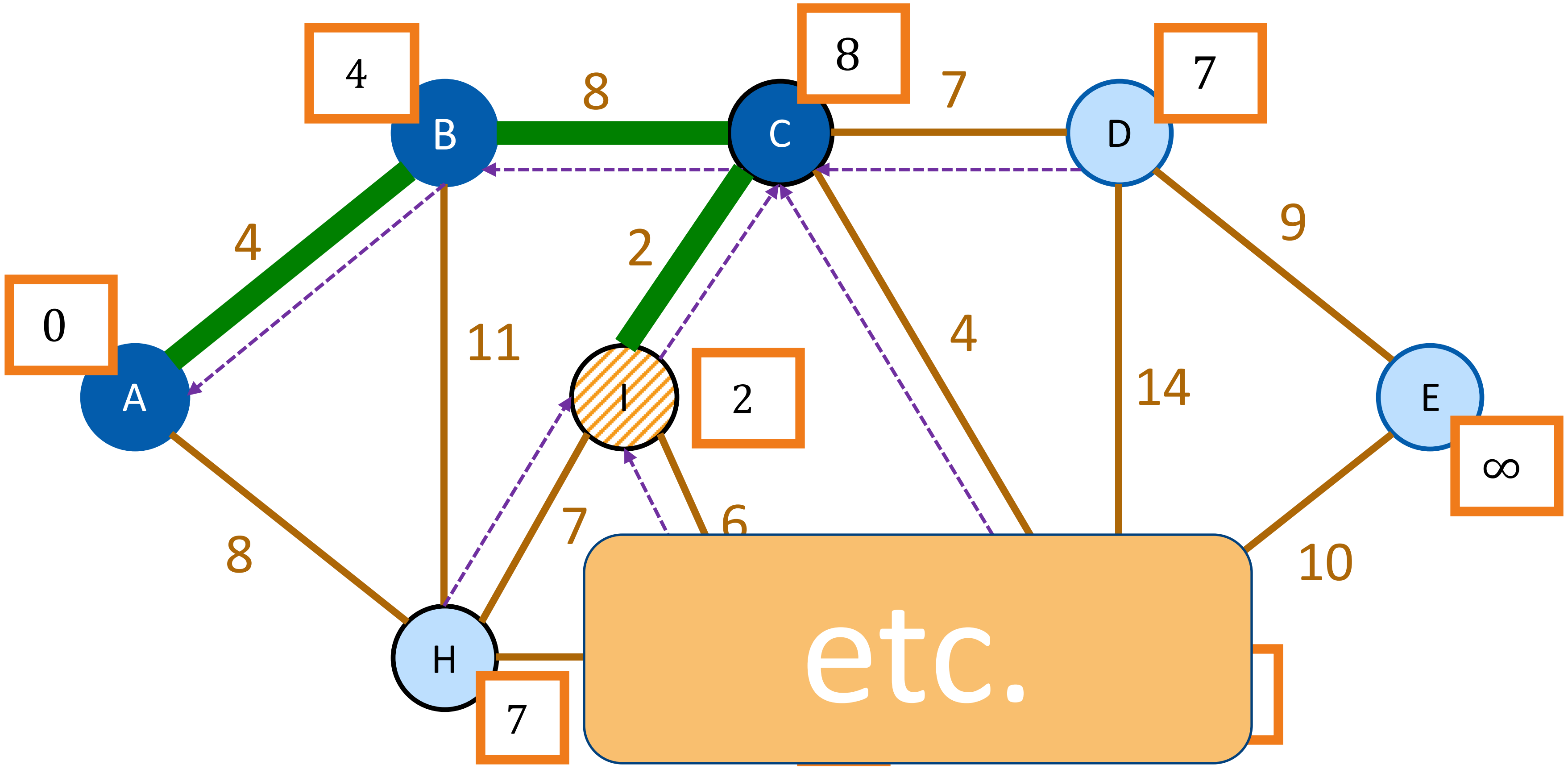

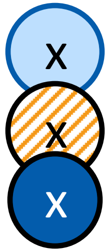

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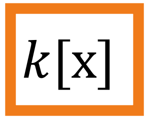

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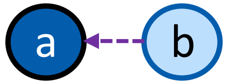

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  - k[v] = min(k[v], weight(u,v))
  - if k[v] updated, p[v] = u

Prim's Algorithm

- Very similar to Dijkstra's algorithm!
- **Differences:**
  1. Keep track of p[v] in order to return a tree at the end
     - But Dijkstra's can do that too, that's not a big difference.
  2. Instead of d[v] which we update by
     - d[v] = min(d[v], d[u] + w(u,v))

     we keep k[v] which we update by
     - k[v] = min(k[v], w(u,v))

Thing 2 is the main difference.

Prim's Algorithm

Two questions

1. Does it work?
   - That is, does it actually return a MST?
     - **YES!**
2. How do we actually implement this?
   - the pseudocode above says "slowPrim"…
     - **Implement it basically the same way we'd implement Dijkstra!**

That's not the only greedy algorithm for MST!

Thoughts

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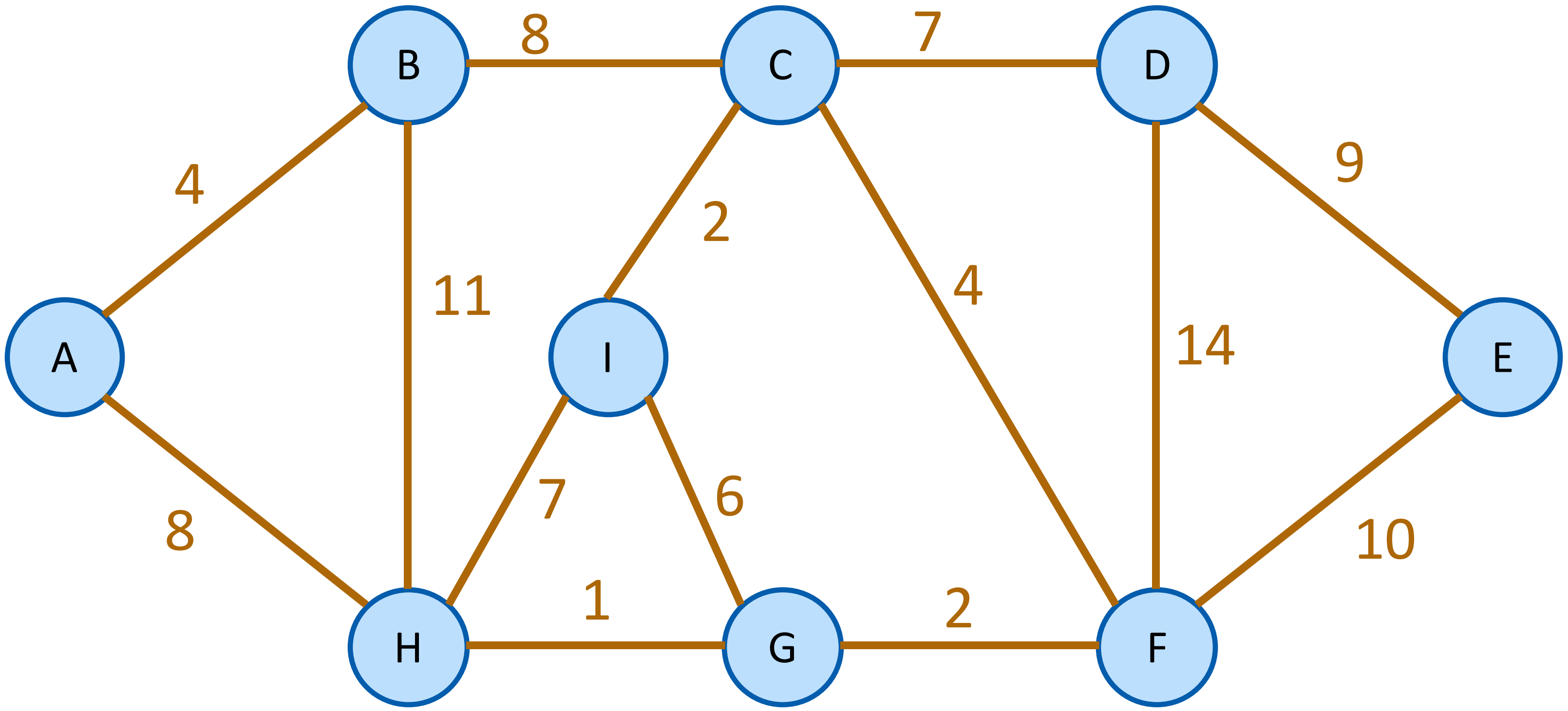
what if we just always take the cheapest edge?
whether or not it's connected to what we have so far?



Thoughts

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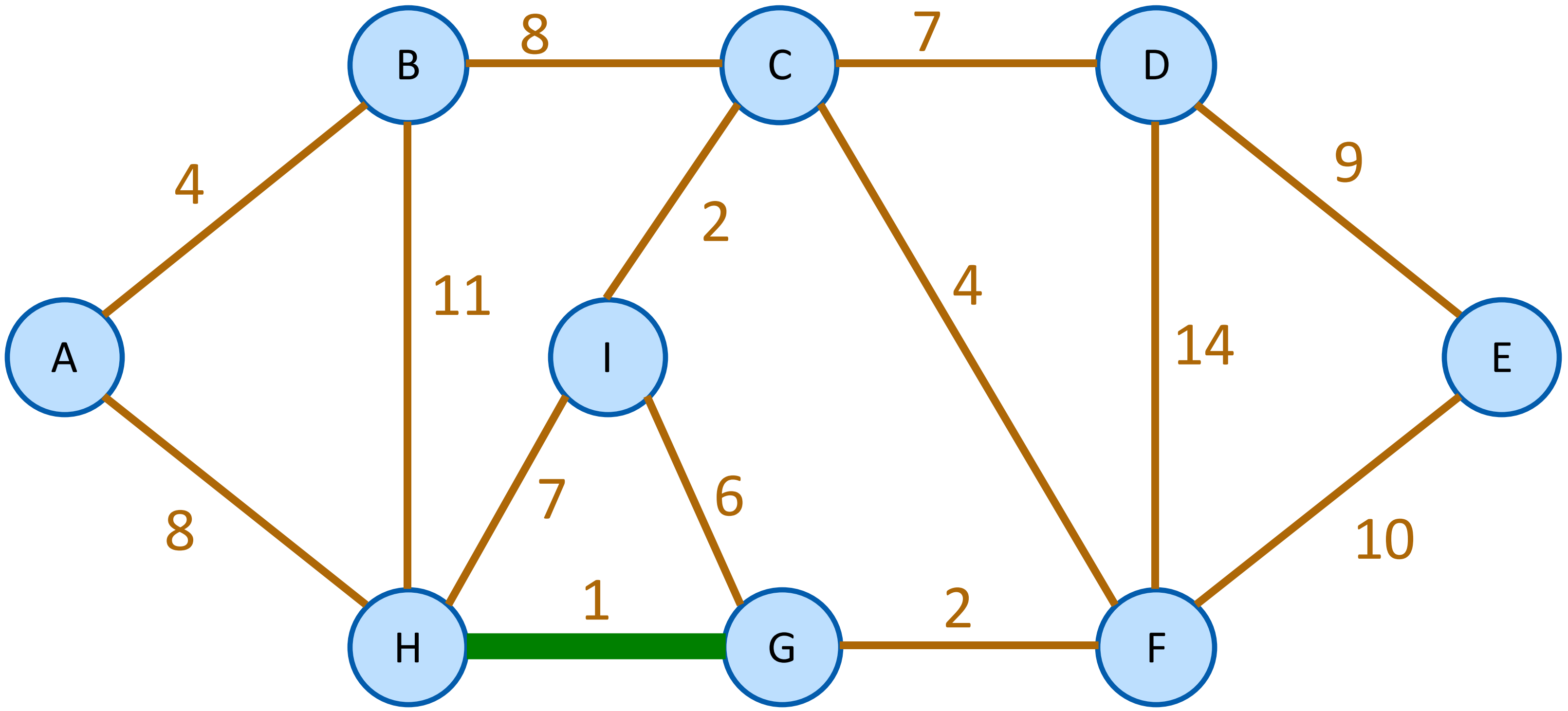

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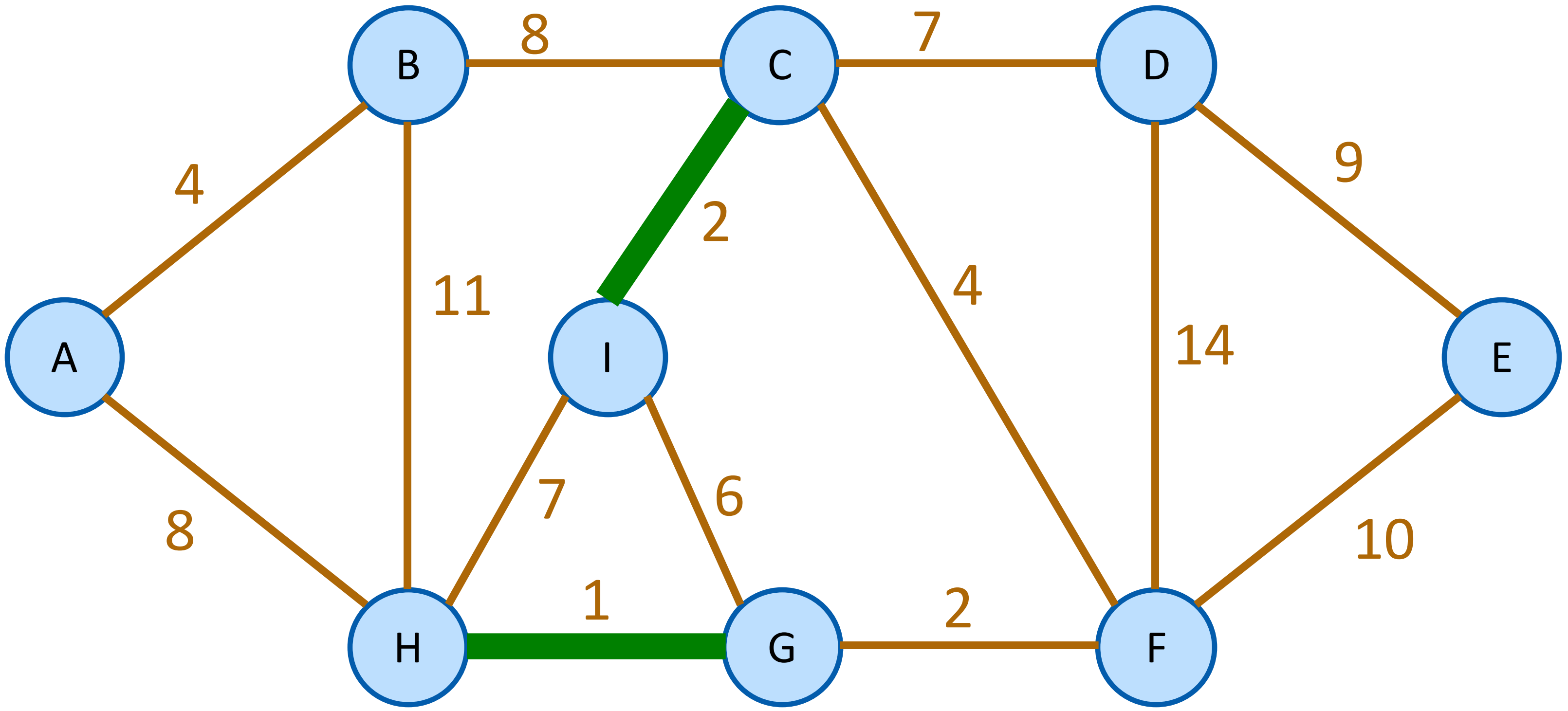

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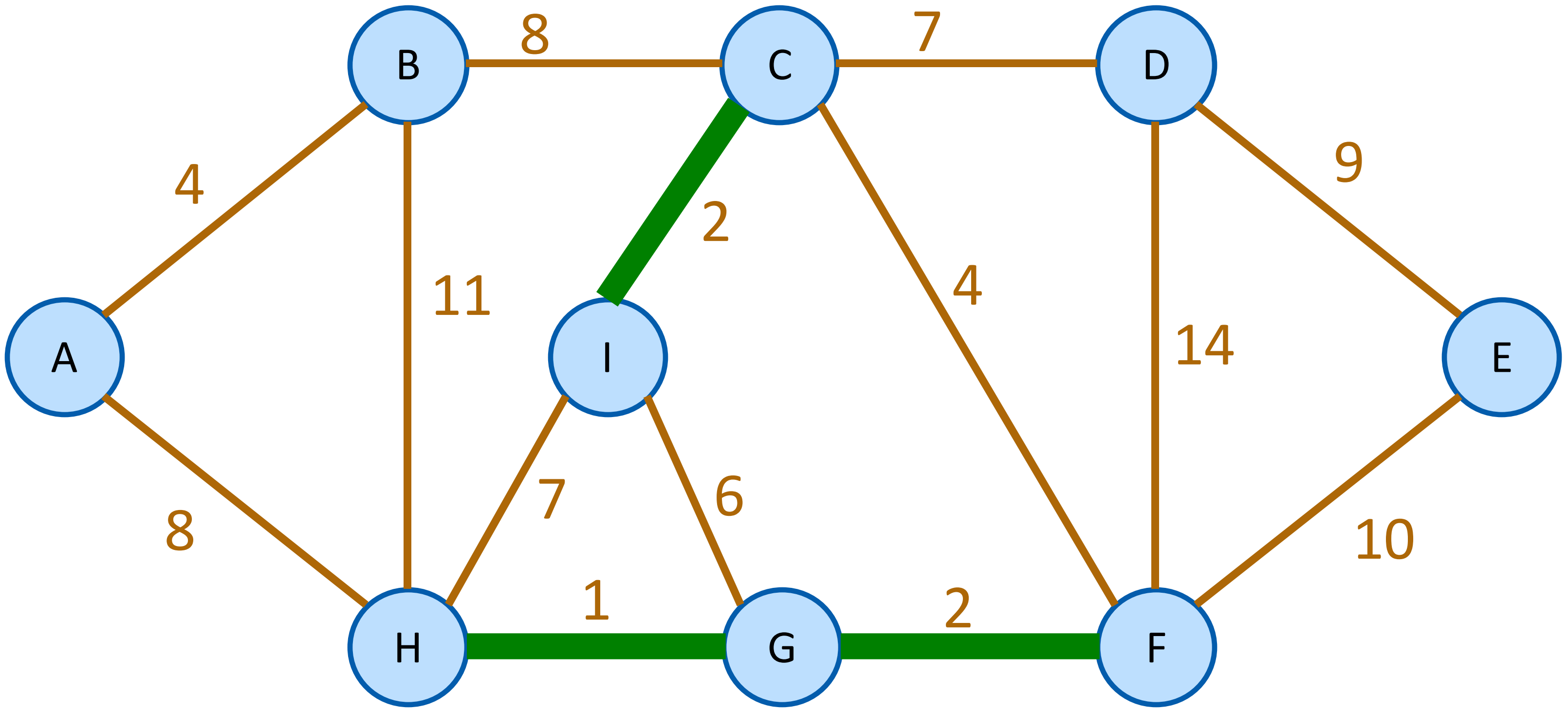

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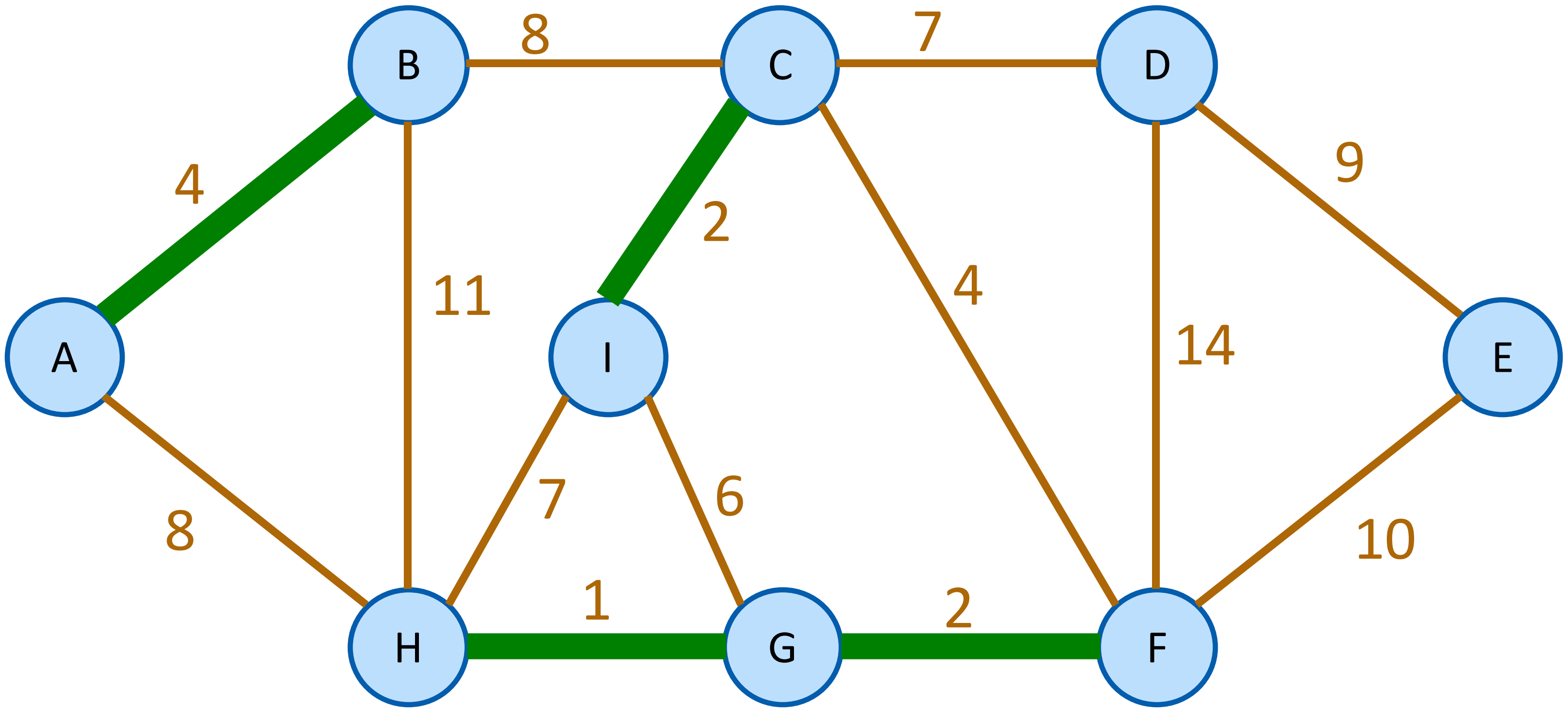

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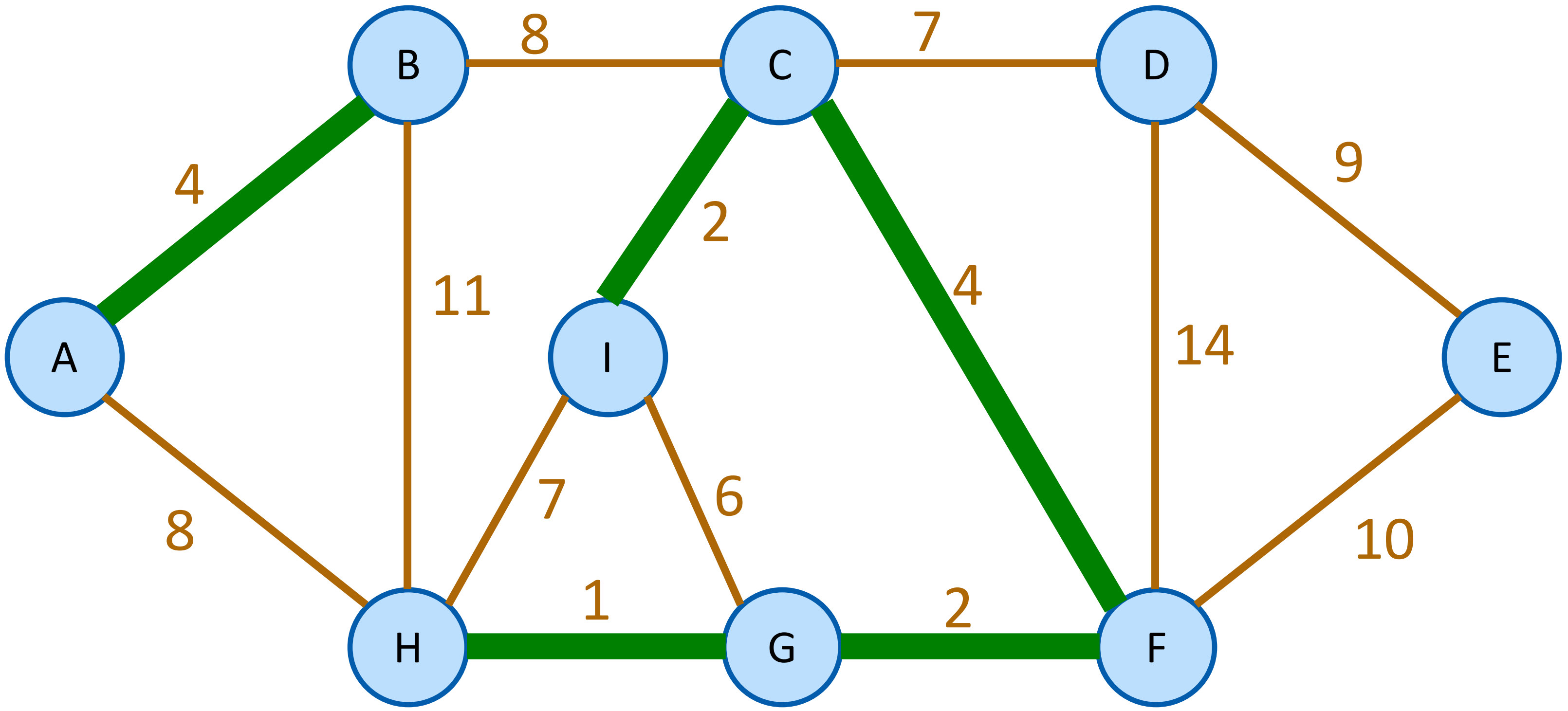

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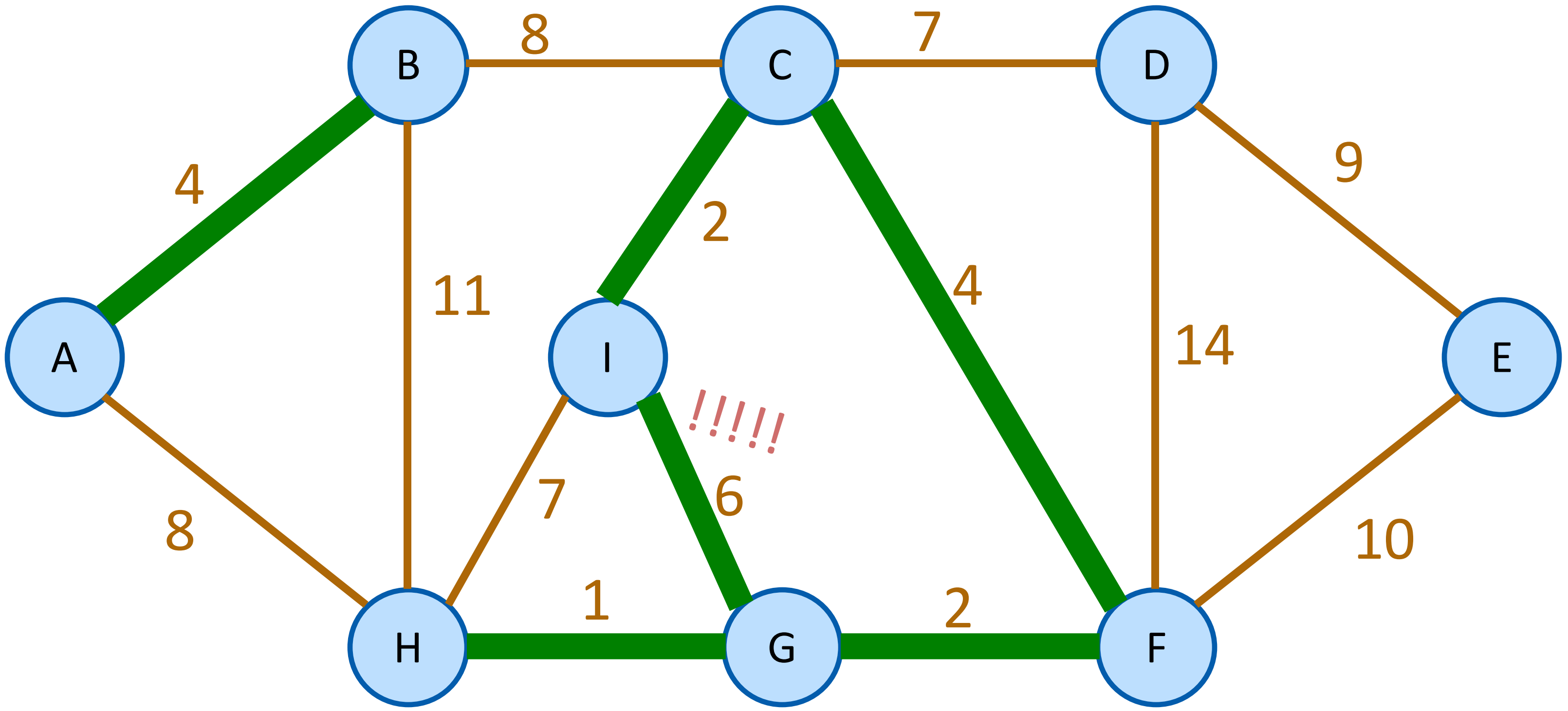

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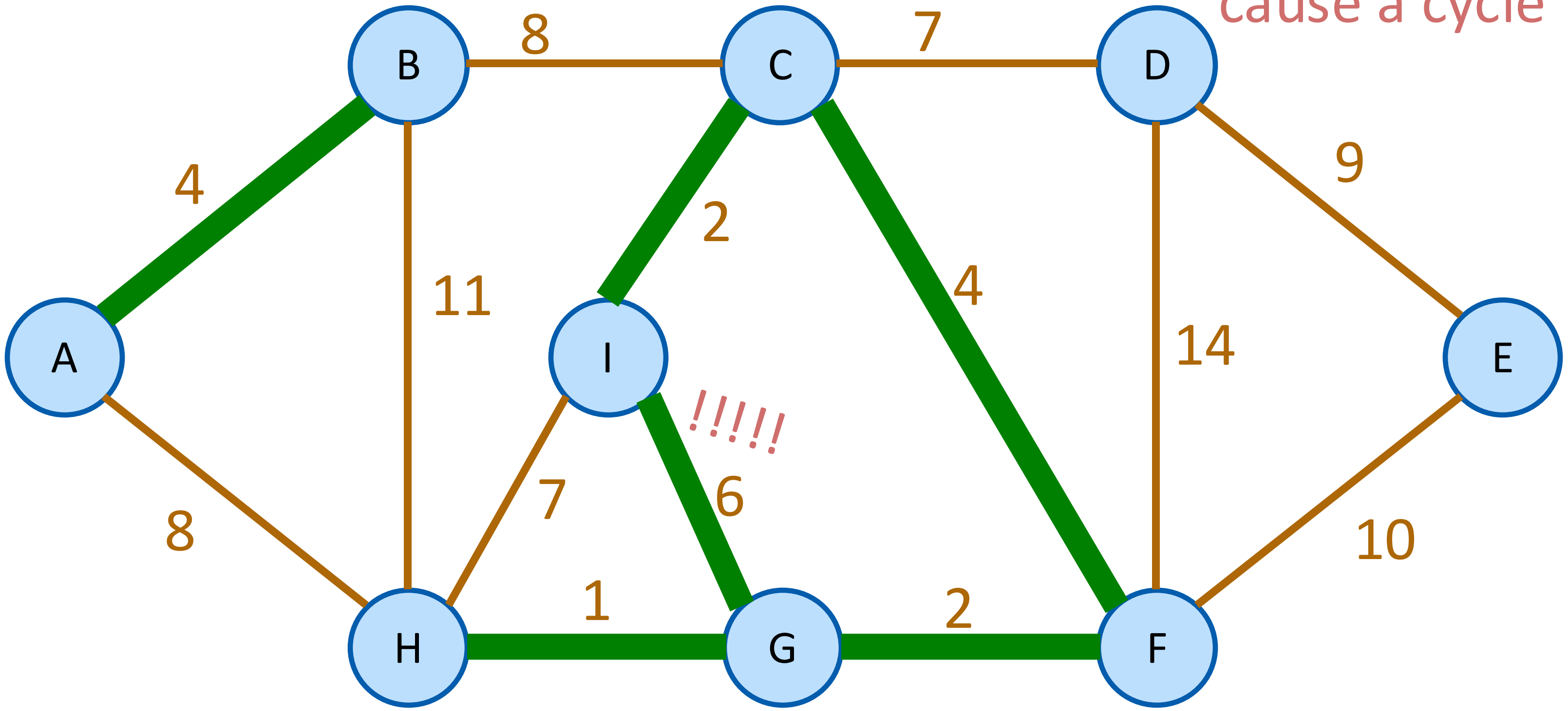
That won't cause a cycle



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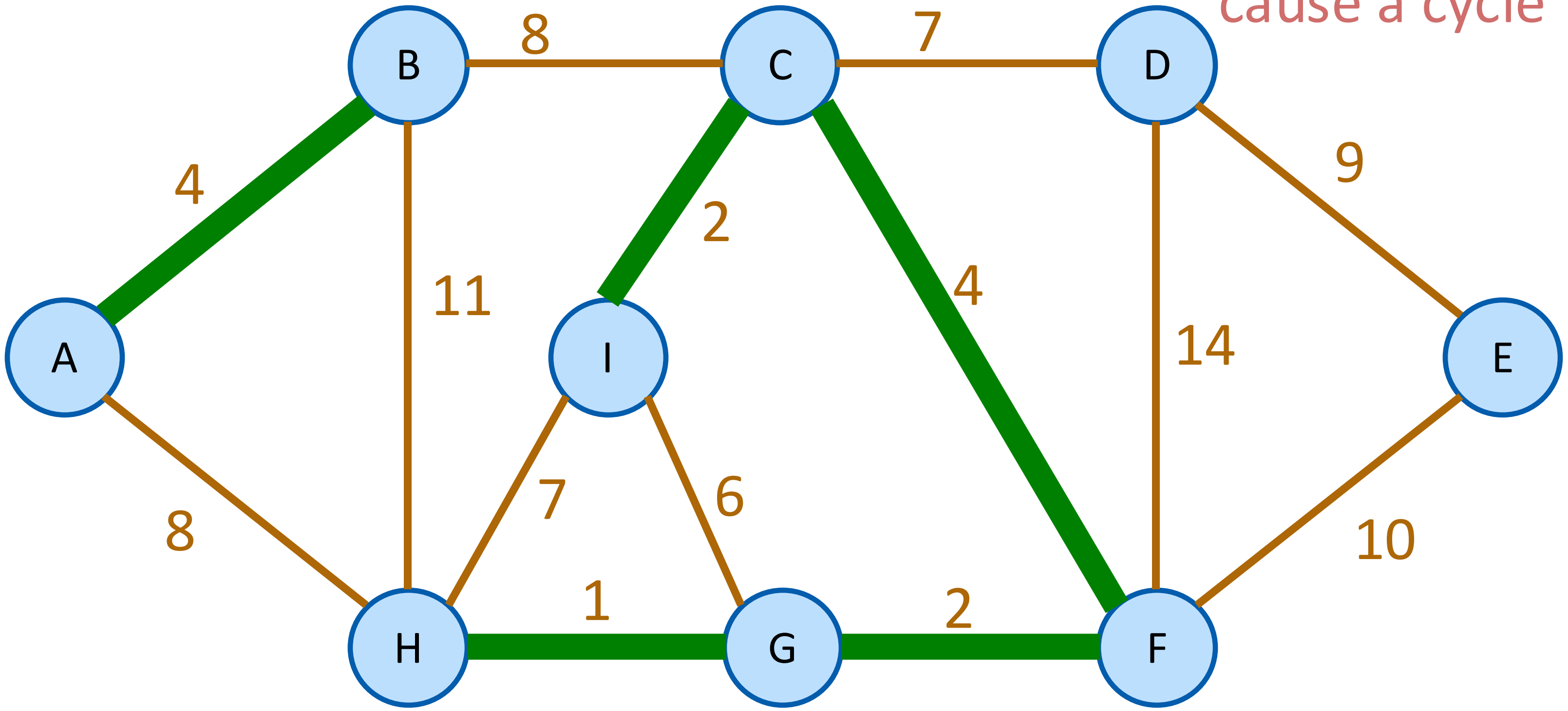

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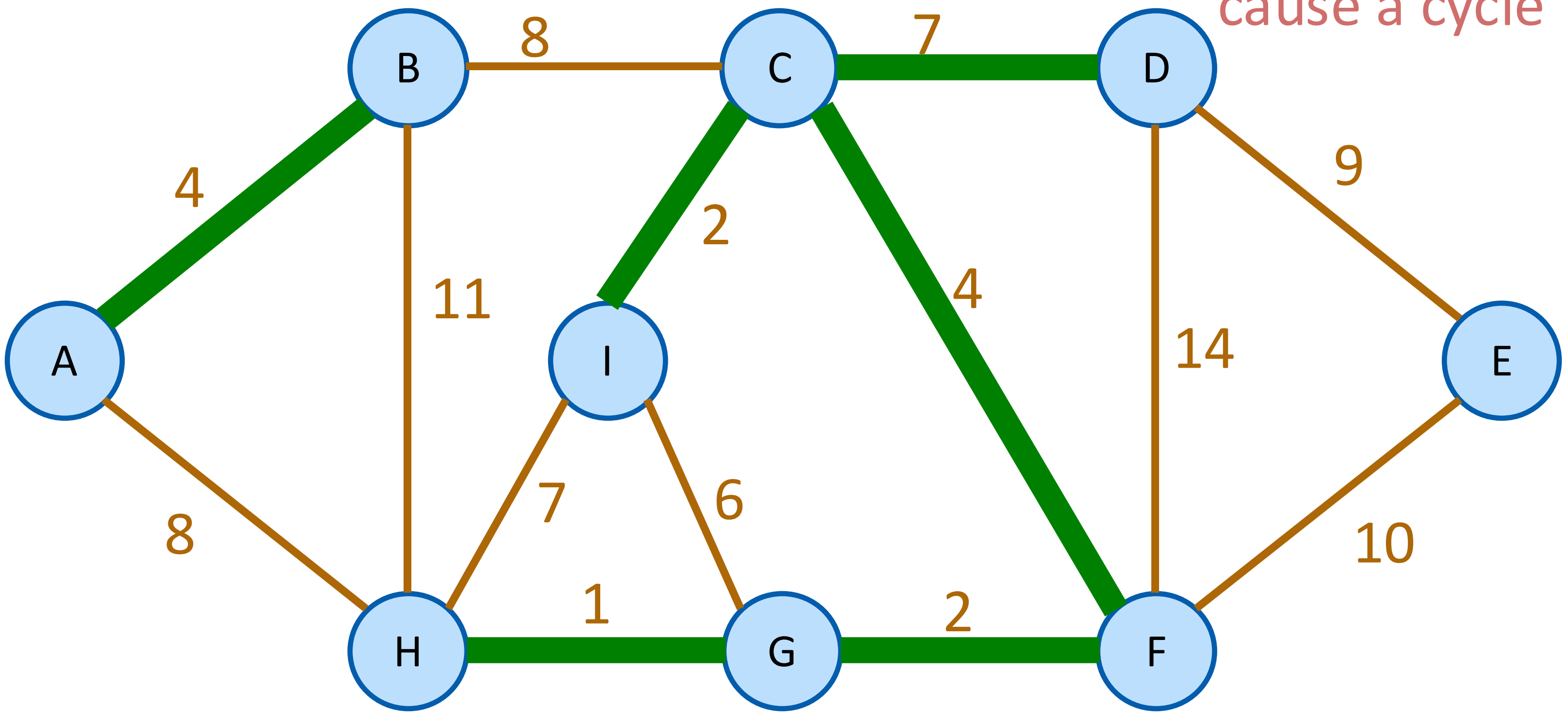

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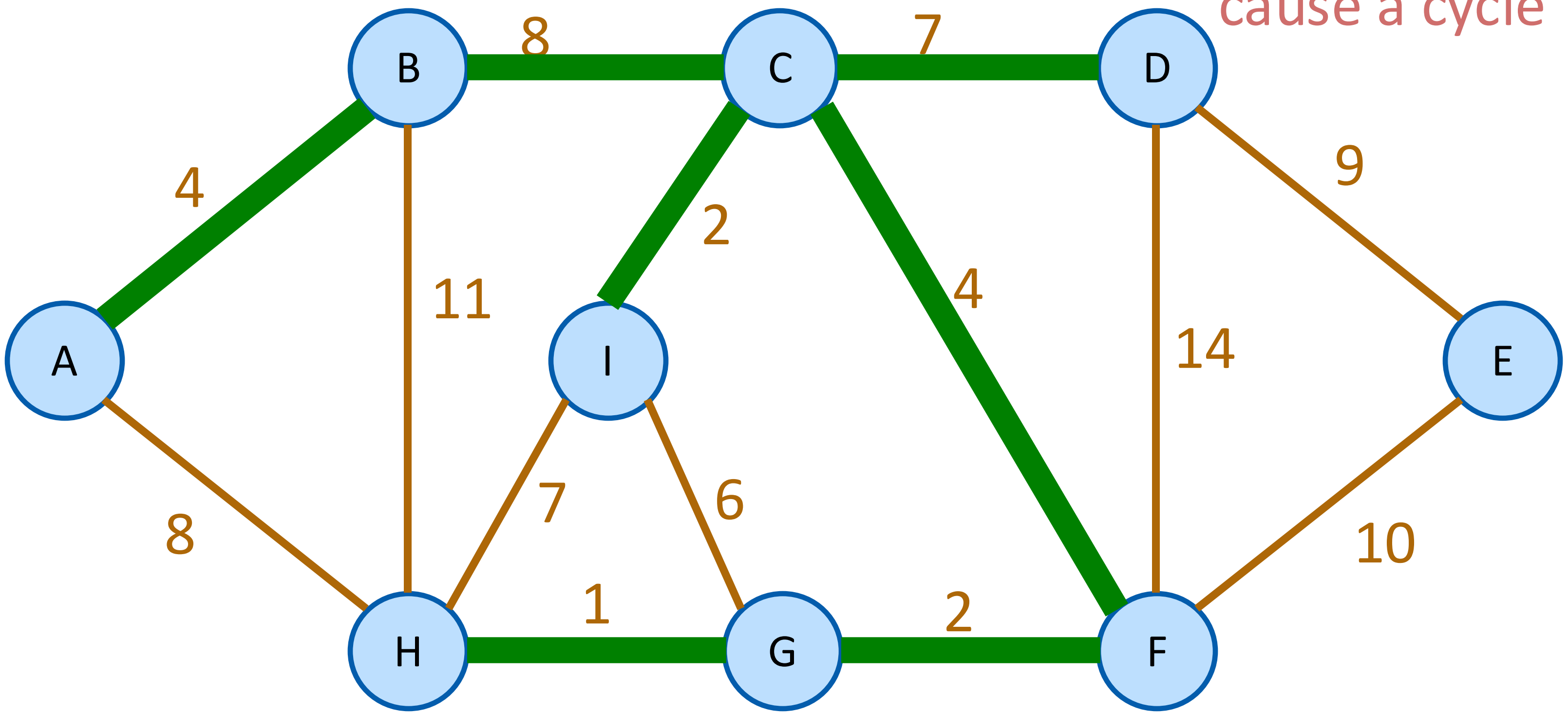

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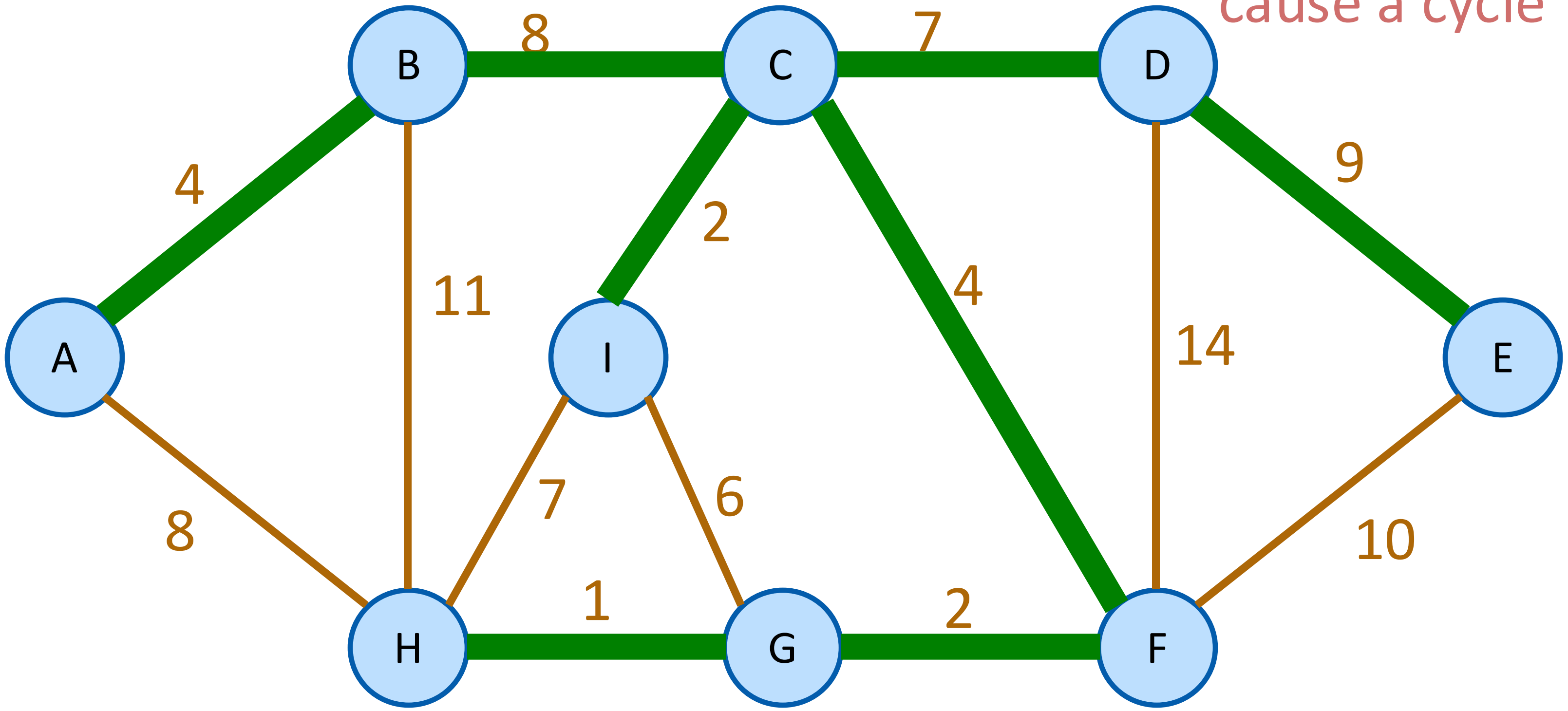


Kruskal's Algorithm

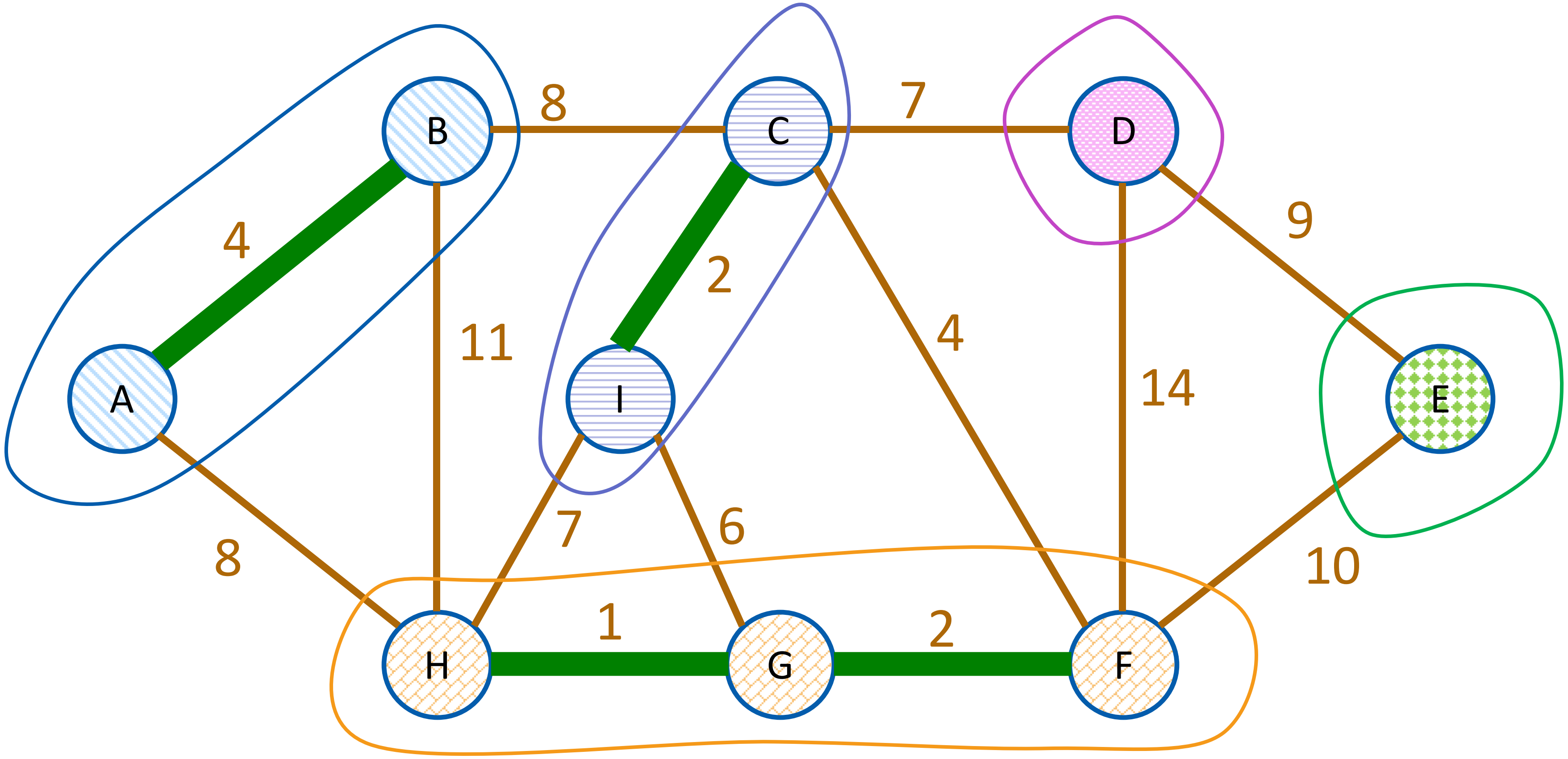
- **slowKruskal**(G = (V,E)):
  - Sort the edges in E by non-decreasing weight.
  - MST = {}
  - **for** e in E (in sorted order):
    - **if** adding e to MST won't cause a cycle:
      - add e to MST.
  - **return** MST

m iterations through this loop

How do we check this?

Kruskal's Algorithm

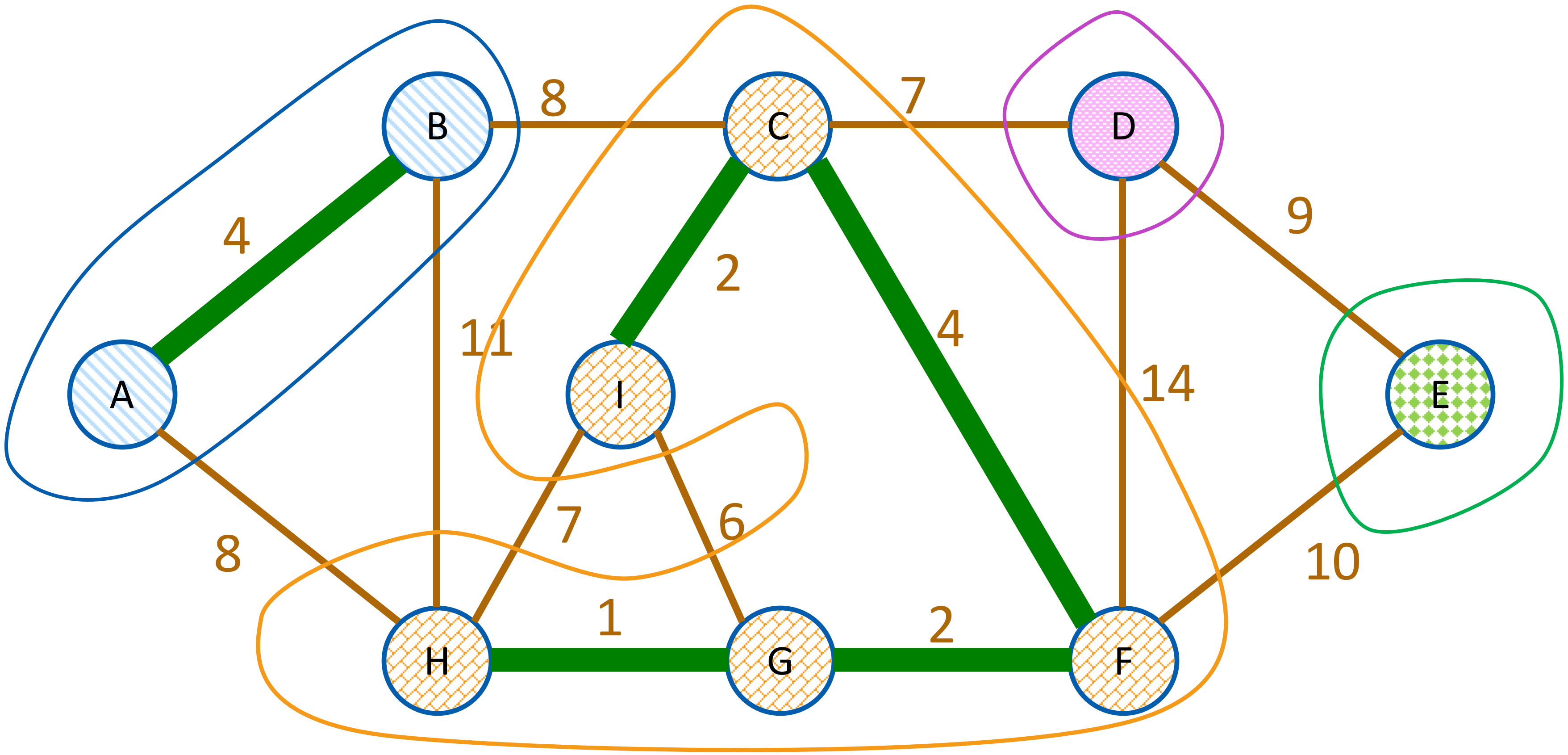
At each step of Kruskal's, we are maintaining a forest.



Kruskal's Algorithm

At each step of Kruskal's, we are maintaining a forest.

When we add an edge, we merge two trees:

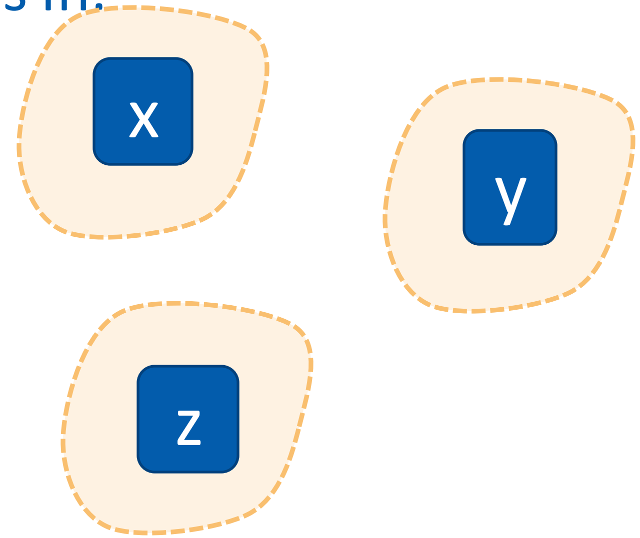


Union-find data structure

- Used for storing collections of sets
- Supports:
  - **makeSet(u):** create a set {u}
  - **find(u):** return the set that u is in
  - **union(u,v):** merge the set that u is in with the set that v is in.

```
makeSet(x)
makeSet(y)
makeSet(z)

union(x,y)
```



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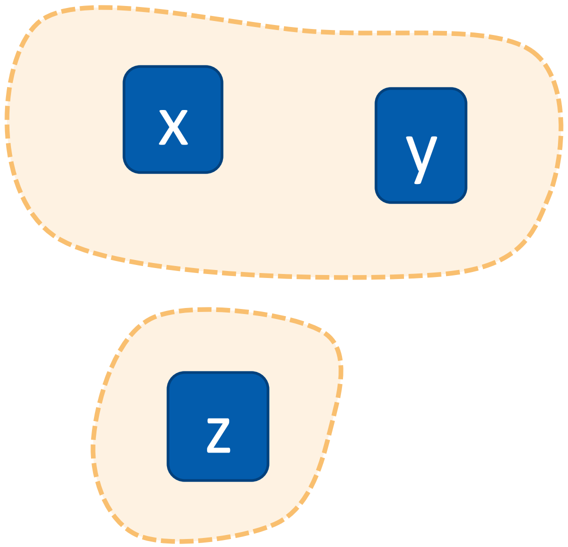

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makeSet(y)
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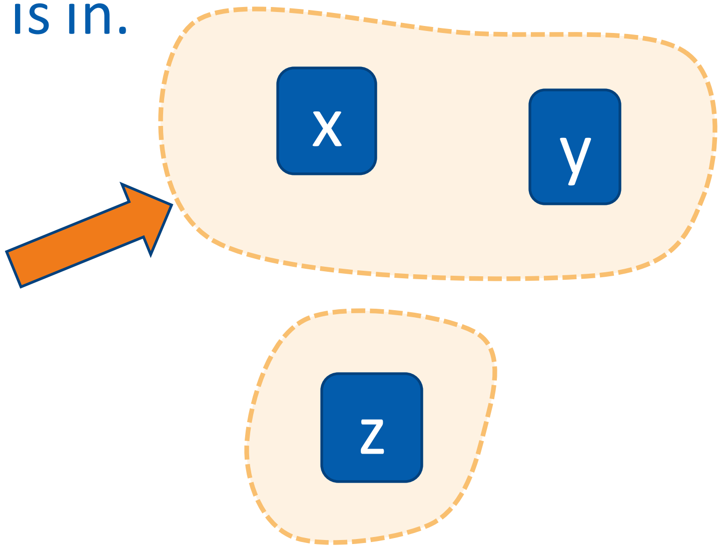
union(x,y)
find(x)
```



Kruskal's Algorithm

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- **kruskal**(G = (V,E)):
  - Sort E by weight in non-decreasing order
  - MST = {} // initialize an empty tree
  - **for** v in V:
    - **makeSet**(v) // put each vertex in its own tree in the forest
  - **for** (u,v) in E: // go through the edges in sorted order
    - **if find**(u) != **find**(v): // if u and v are not in the same tree
      - add (u,v) to MST
      - **union**(u,v) // merge u's tree with v's tree
  - **return** MST

Kruskal's Algorithm

Running time

- Sorting the edges takes O(m log(n))
  - In practice, if the weights are small integers we can use radixSort and take time O(m)
- For the rest:
  - n calls to **makeSet**
    - put each vertex in its own set
  - 2m calls to **find**
    - for each edge, **find** its endpoints
  - n-1 calls to **union**
    - we will never add more than n-1 edges to the tree,
    - so we will never call **union** more than n-1 times.
- Total running time: **O(mlog(n))**

O(1)

$O(\alpha(n))$, amortized

$O(\alpha(n))$, amortized

α(n) is the inverse Ackermann function (grows extremely slowly)

- For $n \leq 2^{65536}$: $\alpha(n)=4$

Kruskal's Algorithm

Does it work?

Leave for your assignment.

Comparison

Are they greedy algorithms?

- Prim:
  - Grows a tree.
  - Time O(mlog(n)) with a red-black tree
  - Time O(m + nlog(n)) with a Fibonacci heap
- Kruskal:
  - Grows a forest.
  - Time O(mlog(n)) with a union-find data structure
  - If you can do radixSort on the weights, morally "O(m)"

Prim might be a better idea on dense graphs if you can't radixSort edge weights

Kruskal might be a better idea on sparse graphs if you can radixSort edge weights

Comparison

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Are they greedy algorithms? YES, BOTH

- Prim:
  - Grows a tree.
  - Time O(mlog(n)) with a red-black tree
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Node centric

Prim might be a better idea on dense graphs if you can't radixSort edge weights

- Kruskal:
  - Grows a forest.
  - Time O(mlog(n)) with a union-find data structure
  - If you can do radixSort on the weights, morally "O(m)"

Edge centric

Kruskal might be a better idea on sparse graphs if you can radixSort edge weights

Can we do better?

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- Karger-Klein-Tarjan 1995:
  - O(m) time randomized algorithm
- Chazelle 2000:
  - O(m· $\alpha(n)$) time deterministic algorithm
- Pettie-Ramachandran 2002:
  - O(The optimal number of comparisons you need to solve the problem, whatever that is…) time deterministic algorithm